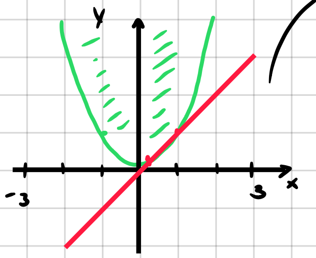
2/6

INTEGRALES MÚLTIPLES

INTEGRALES DOBLES

* Ejercicio 1 : Graficar y calcular áreas reg planos

(a) $y \geq x^2$, $y \leq x$



tipo región 1 (x estático)

↳ hallar D límites de integración → $D = \{(x,y) / y \geq x^2,\ y \leq x,\ 0 \leq x \leq 1,\ x^2 \leq y \leq x\}$

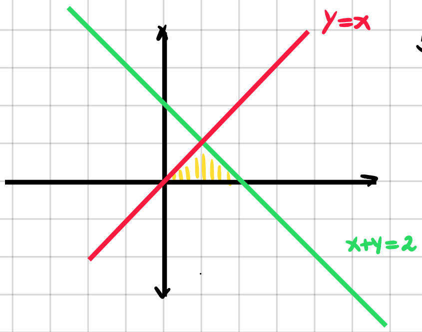
↳ funciones $y_1(x) = x^2$ $y_2(x) = x$

$$\text{área}(D) = \iint_D f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} dy\ dx = \int_0^1 \left(y \Big|_{x^2}^{x} \right) dx = \int_0^1 (x - x^2)\,dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(b) $x + y \leq 2$, $y \leq x$, $y \geq 0$

→ Gráfico

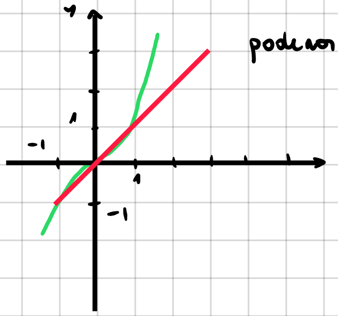


se puede pensar como región tipo 2 (var y fija)

$D = \{(x,y) \mid 0 \leq y \leq 1,\ y \leq x \leq 2-y\}$

$$\text{área}(D) = \int_0^1 dy \cdot \int_y^{2-y} dx = \int_0^1 (2-y) - y\ dy = 2y - y^2 \Big|_0^1 = 1$$

(c) $y = x^3$, $y = x$



podemos pensar región D como una suma $D = D_1 + D_2$, siendo D_1 y D_2 regiones tipo 1

por lo que $\text{área}(D) = \text{área}(D_1) + \text{área}(D_2)$

$D_1 = \{(x,y) \mid -1 \leq x \leq 0,\ x \leq y \leq x^3\}$

$D_2 = \{(x,y) \mid 0 \leq x \leq 1,\ x^3 \leq y \leq x\}$

$$\text{área}(D_1) = \int_{-1}^{0} dx \int_x^{x^3} dy = \int_{-1}^{0} x^3 - x\ dx = \frac{x^4}{4} - \frac{x^2}{2} \Big|_{-1}^{0} = -\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}$$

$$\text{área}(D_2) = \int_0^1 dx \int_{x^3}^{x} dy = \int_0^1 x - x^3\ dx = \frac{x^2}{2} - \frac{x^4}{4} \Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

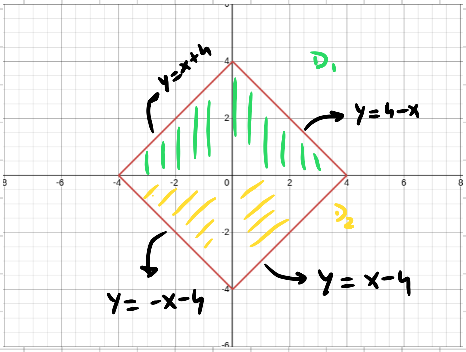
$$\text{area}(D) = \text{area}(D_1) + \text{area}(D_2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

ⓓ Limitado N_4 de $f(x,y) = |x| + |y|$

$y = x+4$

$y = 4 - x$

↳ Graficar $|x| + |y| = 4$



$$D_1 = \{(x,y) \,/\, 0 \le y \le 4,\ y-4 \le x \le 4-y\}$$

$$D_2 = \{(x,y) \,/\, -4 \le y \le 0,\ -4-y \le x \le y+4\}$$

$$\text{area}(D_1) = \int_0^4 \int_{y-4}^{4-y} dx\,dy = \int_0^4 (4-y)-(y-4)\,dy = 8y - y^2 \Big|_0^4 = 16$$

$$\text{area}(D_2) = \int_{-4}^{0} \int_{-4-y}^{y+4} dx\,dy = \int_{-4}^{0} 2y+8\,dy = y^2 + 8y \Big|_{-4}^{0} = -\Big(\underbrace{16-32}_{-16}\Big) = 16$$

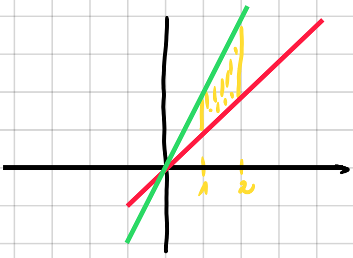
$$\text{area}(D) = \text{area}(D_1) + \text{area}(D_2) = 16 + 16 = 32$$

---

* Ejercicio 2: Graficar región de integración, expr integral invirtiendo orden integración

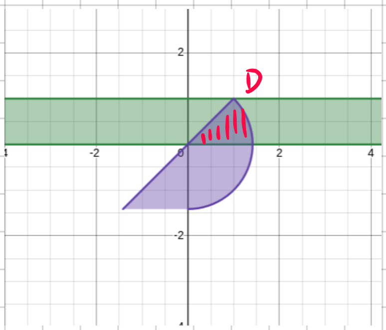
ⓐ $\int_1^2 dx \int_x^{2x} f(x,y)\,dy$ → región tipo 1 (x const)

$$D = \{(x,y) \,/\, 1 \le x \le 2,\ x \le y \le 2x\}$$

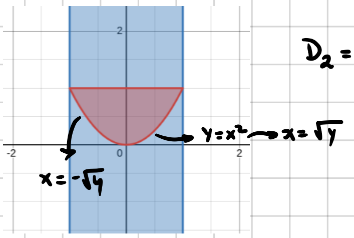


no se puede invertir orden integración

ⓑ $\int_0^1 \int_y^{\sqrt{2-y^2}} f(x,y)\,dx\,dy$ → $D = \{(x,y) \,/\, 0 \le y \le 1,\ y \le x \le \sqrt{2-y^2}\}$



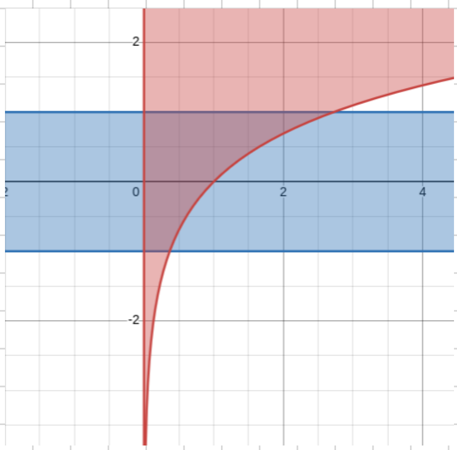
ⓒ $\int_{-1}^{1} \int_{x^2}^{1} f(x,y)\,dy\,dx$ → $D = \{(x,y) \,/\, -1 < x < 1,\ x^2 < y < 1\}$



$$D_2 = \{(x,y) \,/\, 0 \le y \le 1,\ -\sqrt{y} \le x \le \sqrt{y}\}$$

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy$$

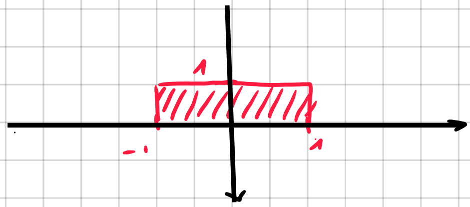
ⓓ $\int_{-1}^{1} dy \int_0^{e^y} f(x,y)\,dx$ $\quad D = \{(x,y) \,/\, -1 \le y \le 1,\ 0 \le x \le e^y\}$



$D^* = \{$

* Ejercicio 3 hallar centro de masa

→ gráfico placa plana



⇒ Datos enunciado

↳ placa plana : $|x| \leq 1$, $0 \leq y \leq 1$; $|x| \leq 1 \to -1 < x < 1$

↳ Densidad = $\delta(x,y) = k|y|$ → ojo es $k|x|$ (eje y)

⇒ Hallar masa total

$$\int_{-1}^{1}\int_{0}^{1} k|y|\,dy\,dx = \int_{-1}^{1}\left(\frac{ky^2}{2}\Big|_0^1\right)dx = \int_{-1}^{1}\frac{k}{2}\,dx = \frac{kx}{2}\Big|_{-1}^{1} = \frac{k}{2}+\frac{k}{2} = k$$

$$\int_0^1\int_{-1}^{1} k|y|\,dx\,dy = \int_0^1\left(k|y|x\Big|_{-1}^{1}\right)dy = \int_0^1 (k|y|+k|y|)\,dy = ky^2\Big|_0^1 = k$$

⇒ Hallar centro de masa = (x_{cm}, y_{cm})

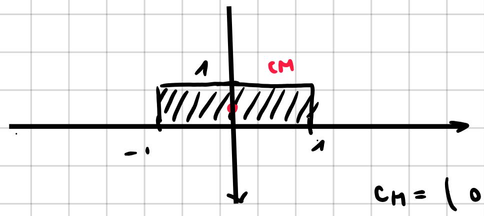
$$x_{cm} = \frac{1}{k}\int_{-1}^{1}\int_0^1 k|y|x\,dy\,dx = \int_{-1}^{1}\int_0^1 |y|x\,dy\,dx = \int_{-1}^{1}\left(\frac{y^2}{2}x\Big|_0^1\right)dx = \int_{-1}^{1}\left(\frac{x}{2}\right)dx = \frac{x^2}{4}\Big|_{-1}^{1} = \frac{1}{4}-\frac{1}{4} = 0$$

$$y_{cm} = \frac{1}{k}\int_{-1}^{1}\int_0^1 k\,y^2\,dy\,dx = \int_{-1}^{1}\int_0^1 y^2\,dy\,dx = \int_{-1}^{1}\left(\frac{y^3}{3}\Big|_0^1\right)dx = \frac{1}{3}\int_{-1}^{1}dx = \frac{x}{3}\Big|_{-1}^{1} = \frac{1}{3}-\left(-\frac{1}{3}\right) = \frac{2}{3}$$

$\boxed{3/6}$

* Ejercicio 3 (rehacer)

→ gráfico placa plana



↳ $\delta(x,y) = k|x|$

↳ placa plana $|x| \leq 1$, $0 \leq y \leq 1$

$-1 < x < 1$

CM = $\left(0, \frac{1}{2}\right)$
(x_g, y_g)

masa placa plana

$$\text{masa} = \int_{-1}^{1}dx\int_0^1 \delta(x,y)\,dy = \int_{-1}^{1}dx\int_0^1 k|x|\,dy = \int_{-1}^{1}\left(k|x|y\Big|_0^1\right)dx = \int_{-1}^{1}k|x|\,dx = \int_{-1}^{0}k\cdot -x\,dx + \int_0^1 kx\,dx$$

$$= \left(-\frac{kx^2}{2}\Big|_{-1}^{0}\right) + \left(\frac{kx^2}{2}\Big|_0^1\right) = \left(0-\left(\frac{-k}{2}\right)\right)+\frac{k}{2} = k$$

Hallar centro de masa

$$x_G = \frac{1}{k}\int_{-1}^{1}dx\int_0^1 k|x|x\,dy = \int_{-1}^{1}|x|x\,dx = \int_{-1}^{0}-x^2\,dx + \int_0^1 x^2\,dx = \left(-\frac{x^3}{3}\Big|_{-1}^{0}\right)+\left(\frac{x^3}{3}\Big|_0^1\right)$$

$$\left(0-\left(-\frac{(-1)^3}{3}\right)\right) + \frac{1}{3} = 0$$

$$y_G = \frac{1}{k}\int_{-1}^{1}dx\int_0^1 k|x|y\,dy = \frac{1}{k}\int_{-1}^{1}\left(k|x|\frac{y^2}{2}\Big|_0^1\right)dx = \frac{1}{k}\int_{-1}^{1}\frac{k|x|}{2}\,dx = \int_{-1}^{1}\frac{|x|}{2}\,dx = \int_{-1}^{0}\frac{-x}{2}\,dx + \int_0^1\frac{x}{2}\,dx$$

$$\left(\frac{-x^2}{4}\Big|_{-1}^{0}\right)+\left(\frac{x^2}{4}\Big|_0^1\right) = \left(0-\left(-\frac{1}{4}\right)\right)+\left(\frac{1}{4}-0\right) = \frac{1}{2}$$

8/6

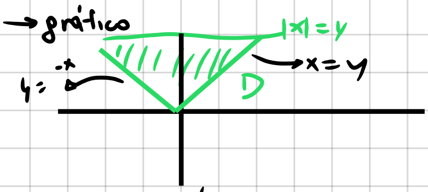
* Ejercicio 4 = Hallar masa de placa plana

⇒ Datos enunciado

↳ placa = $|x| \le y \le 2$

↳ densidad $\delta(x,y) = k\,|y|$

→ gráfico



$$D = \{(x,y) / \ 0 \le y \le 2 \ , \ -y \le x \le y\}$$

$$|x-1| = \begin{cases} x-1 & \text{si } x>1 \\ 1-x & \text{si } x<0 \end{cases}$$

⇒ Hallar masa total

$$\int_0^2 dy \int_{-y}^{y} \delta(x,y)\,dx = \int_0^2 dy \int_{-y}^{y} k|x-1|\,dx$$

$$\int_{-y}^{y} k|x-1|\,dx = k\left(\int_{-y}^{1} 1-x\,dx + \int_{1}^{y} x-1\,dx\right) = k\left[\left(x-\frac{x^2}{2}\Big|_{-y}^{1}\right) + \left(\frac{x^2}{2}-x\Big|_{1}^{y}\right)\right]$$

$$= k\left[\left[\left(1-\frac{1}{2}\right)-\left(-y-\frac{y^2}{2}\right)\right] + \left[\left(\frac{y^2}{2}-y\right)-\left(\frac{1}{2}-1\right)\right]\right]$$

$$= k\left[\left(\frac{1}{2}+y+\frac{y^2}{2}\right)+\left(\frac{y^2}{2}-y+\frac{1}{2}\right)\right]$$

$$k\left[y^2+1\right]$$

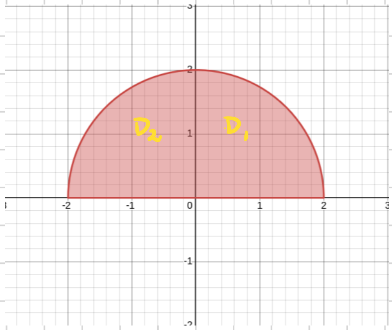
$$\int_0^2 dy \int_{-y}^{y} k|x-1|\,dx = k\int_0^2 y^2+1\,dy = k\left(\frac{y^3}{3}+y\right)\Big|_0^2 = k\left(\frac{8}{3}+2\right) = k\,\frac{14}{3}$$

está mal hecho zona de corte

si $|x-1|$ es módulo entonces $|x-1| = \begin{cases} x-1 & x>1 \\ 1-x \text{ si } x<1 \end{cases}$

* Ejercicio 5 : Calcular densidad media $= \dfrac{\text{masa total}}{\text{area total}}$

• gráfico placa plana



↳ placa definida : $0 \le y \le \sqrt{4-x^2}$

↳ $\delta(x,y) = k|x|$, $k>0$

⇒ Observación

- la región $D = \{(x,y) \in \mathbb{R}^2 / 0 \le y \le \sqrt{4-x^2}\}$ se puede dividir

en dos regiones D_1 y D_2 iguales, tales que

$D = D_1 + D_2 = 2D_1 = 2D_2$. Esto implica que $\text{Area}(D) = 2 \cdot \text{Area}(D_1)$

y $\text{masa}(D) = 2 \cdot \text{masa}(D_1)$

⇒ Establecer D_1

$D_1 = \{(x,y) \in \mathbb{R}^2 / 0 \le x \le 2 \ \wedge \ 0 \le y \le \sqrt{4-x^2}\} \longrightarrow \{(x,y) / 0 \le y \le 2 , \ 0 \le x^2 \le \sqrt{4-y^2}$

$D_{r\theta} = \begin{cases} x = r\cos(\theta) \\ y = r\,\text{sen}(\theta) \end{cases}$. $0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$

⇒ Hallar área Total

$$\text{área}(D_1) = \int_0^2 \int_0^{\sqrt{4-x^2}} dy\,dx = \int_0^2 \left(y \Big|_0^{\sqrt{4-x^2}} \right) dx = \int_0^2 \sqrt{4-x^2}\,dx = \int_0^2 \sqrt{(2+x)(2-x)}\,dx \ ?$$

$$\text{área}(D_1) = \int_0^{\frac{\pi}{2}} \int_0^2 r \cdot dr\,d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{r^2}{2} \Big|_0^2 \right) d\theta = \int_0^{\frac{\pi}{2}} 2\,d\theta = 2\theta \Big|_0^{\frac{\pi}{2}} = \pi$$

area total $= A(D) = 2A(D_1) = 2\pi$

* Hallar masa total

$$\text{masa}(D_1) = \int_0^2 \int_0^{\sqrt{4-x^2}} k|x|\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^2 k|r\cos(\theta)|\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^2 k r^2 \cos(\theta)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(k\frac{r^3}{3}\cos(\theta) \Big|_0^2 \right) d\theta = \int_0^{\frac{\pi}{2}} k\frac{8}{3}\cos(\theta)\,d\theta = -k \cdot \frac{8}{3}\,\text{sen}\,\theta \Big|_0^{\frac{\pi}{2}} = k\frac{8}{3}\,\text{sen}(\theta) \Big|_0^{\frac{\pi}{2}} = k\frac{8}{3}$$

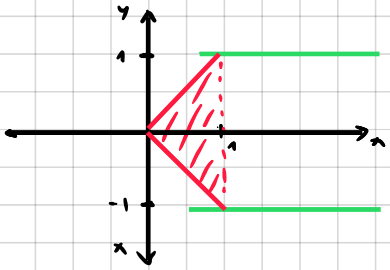
masa total $= 2 \cdot \text{masa}(D_1) = \frac{16k}{3}$

densidad prom $=$ masa tot / area tot $= \frac{16k}{3} / 2\pi = \frac{8k}{3\pi}$

* Ejercicio 7 : Interpretar gráficamente región y calcular integrales

ⓐ $\int_{-1}^{1}\int_{|y|}^{1} 2x \, dx\, dy$ región $D=\{(x,y)\in\mathbb{R}^2 / |y|\le x\le 1 \ , \ -1\le y\le 1\}$

→ gráfico región D



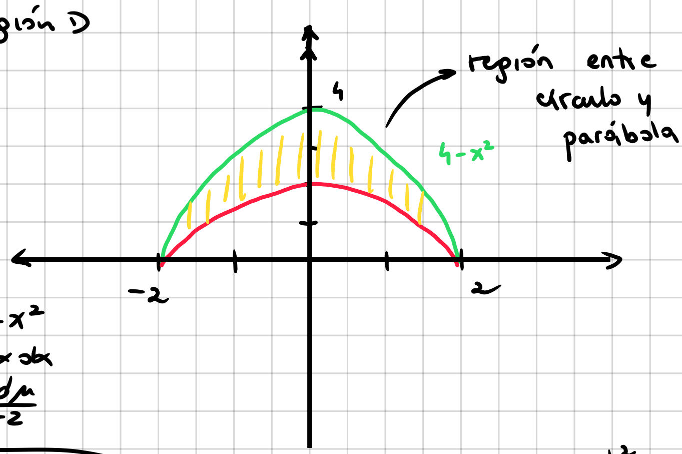
$$\int_{-1}^{1}\int_{|y|}^{1} 2x\, dx\, dy = \int_{-1}^{1}\left(x^2\Big|_{|y|}^{1}\right)dy = \int_{-1}^{1} 1-y^2\, dy$$

$$= y-\frac{y^3}{3}\Big|_{-1}^{1} = \left(1-\frac{1}{3}\right)-\left(-1-\frac{(-1)}{3}\right)$$

$$= \frac{2}{3}-\left(-\frac{2}{3}\right) = \frac{4}{3}$$

ⓑ $\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{4-x^2} x\, dy\, dx$ región $D=\{(x,y)\in\mathbb{R}^2 / -2\le x\le 2 \ \wedge \ \sqrt{4-x^2}\le y\le 4-x^2\}$

└ gráfico región D



$$\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{4-x^2} x\, dy\, dx = \int_{-2}^{2}\left(xy\Big|_{\sqrt{4-x^2}}^{4-x^2}\right)dx$$

$$= \int_{-2}^{2} x(4-x^2) - x\sqrt{4-x^2}\, dx$$

$$= \int_{-2}^{2}\left[(4-x^2)-\sqrt{4-x^2}\right]x\, dx \ ,$$

$\mu = 4-x^2$
$d\mu = -2x\, dx$
$x\,dx = \frac{d\mu}{-2}$

$$= \int \mu-\sqrt{\mu}\, d\mu = \int \mu\, d\mu + \int \mu^{\frac{1}{2}}\, d\mu$$

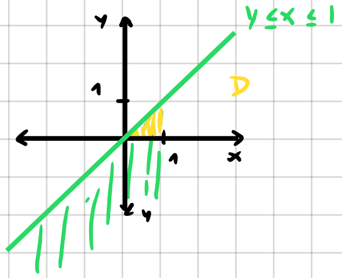
$$= \frac{\mu^2}{2}+\frac{\mu^{\frac{3}{2}}}{\frac{3}{2}} = \frac{\mu^2}{2}+\frac{2\mu^{\frac{3}{2}}}{3}$$

$$= \frac{(4-x^2)}{2}+\frac{2\,(4-x^2)^{\frac{3}{2}}}{3}\Bigg|_{-2}^{2}$$

$$\left[\frac{0}{2}+\frac{2.(4-4)^2}{3}\right]+\left[\frac{0}{2}+0\right]=0$$

ⓒ $\int_0^1\int_y^1 e^{x^2}\, dx\, dy$ región $D=\{(x,y)\in\mathbb{R}^2 / 0\le y\le 1 \wedge y\le x\le 1\}$

→ gráfico región D



$$\underbrace{\int_0^1\int_y^1 e^{x^2}\, dx\, dy}$$ cambiar orden

$D_1=\{(x,y)/ \ 0\le x\le 1, \ 0\le y\le x\}$

$\mu = x^2$ $d\mu = 2x\,dx$
$x\,dx = \frac{d\mu}{2}$

$$\int_0^1\int_0^x e^{x^2}\, dy\, dx = \int_0^1\left(e^{x^2}y\Big|_0^x\right)dx = \int_0^1 e^{x^2}x\, dx = \left(\frac{e^{x^2}}{2}\Big|_0^1\right) = \frac{e}{2}-\frac{1}{2}$$

$$= \frac{1}{2}(e-1)$$

$$\int e^{x^2}x\,dx = \int e^{\mu}\frac{d\mu}{2} = = \frac{e^{\mu}}{2} = \frac{e^{x^2}}{2}$$

Cambios de variables   Integrales dobles

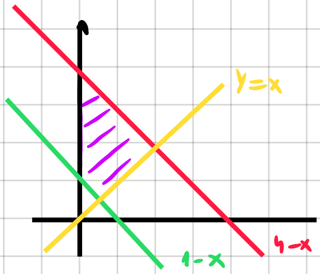
$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(\mu,v)\,|J(\mu,v)|\,d\mu\,dv$$

* Ejercicio 9: Calcular las integrales con TL

ⓐ $\iint_D x\,dx\,dy$, $D=\{(x,y) / \underbrace{1\le x+y\le 4}_{1-x\le y\le 4-x},\ x\ge 0,\ y\ge x\}$

$$\begin{cases} x = v \\ y = \mu - v \end{cases} \qquad \underbrace{x+y=\mu-v+v=\mu}_{\mu = x+y}$$

↳ gráfica región D



↳ cambio de variables $(x,y)=(v,\mu-v)$

↳ transformar región D al transf lineal

→ $1\le x+y\le 4 \longrightarrow 1\le \mu\le 4$

→ $x\ge 0 \longrightarrow v\ge 0$

↳ $y\ge x \longrightarrow \mu - v\ge v \longrightarrow \mu\ge 2v$

$D^* = \{(\mu,v)\in\mathbb{R}^2 / 1\le\mu\le 4,\ 0\le v\le \frac{\mu}{2}\}$

↳ Hallar jacobiano $|J(\mu,v)| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = |-1| = 1$

→ Calcular integral

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(\mu,v)\,|J(\mu,v)|\,d\mu\,dv$$

$$= \int_1^4 \int_0^{\mu/2} v\,dv\,d\mu = \int_1^4 \left(\frac{v^2}{2}\Big|_0^{\mu/2} \right) d\mu = \int_1^4 \frac{\mu^2}{8}\,d\mu = \frac{\mu^3}{24}\Big|_1^4 = \frac{8}{3} - \frac{1}{24} = \frac{21}{8}$$

ⓑ $\iint_D e^{(y-x)/(x+y)}\,dx\,dy$ $x+y\le 2$. $x\ge 0$, $y\ge 0$

↳ Hallar transformación lineal   ↳ Hallar transformación de D

$$\begin{cases} \mu = x+y \\ v = x \end{cases} \qquad \begin{cases} \mu - v = y \\ \mu - 2v = y-x \end{cases}$$

• $x+y\le 2 \rightarrow \mu\le 2$ • $y\ge 0 \rightarrow \mu - v\ge 0 \rightarrow \mu\ge v$

• $x\ge 0 \rightarrow v\ge 0$ $0\le v\le \mu$ $\underbrace{0\le\mu\le 2}_{0\le v\le\mu\le 2}$

→ Jacobiano $|J(\mu,v)| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = |-1| = 1$

→ calcular integral

$$\iint_D e^{(y-x)(x+y)}\,dx\,dy = \int_0^2 \int_0^\mu e^{(\mu-2v)/\mu}\,dv\,d\mu = \int_0^2\int_0^\mu e^{1-\frac{2v}{\mu}}\,dv\,d\mu = \int_0^2 \left(\frac{\mu}{-2} e^{1-\frac{2v}{\mu}} \Big|_0^\mu \right) d\mu$$

$$= \int_0^2 \frac{\mu}{-2e}\,d\mu = \frac{\mu^2}{-4e}\Big|_0^2 = -\frac{1}{e}$$

$$\int e^{1-\frac{2v}{\mu}}\,dv = \int e^t \frac{\mu}{-2}\,dv = \frac{\mu}{-2} e^{1-\frac{2v}{\mu}}$$

$t = 1-\frac{2v}{\mu}$ $dt = \frac{-2}{\mu}\,dv$

ⓒ $\iint_D (x-y)^2 \operatorname{sen}^2(x+y)\, dx\, dy$ D: $-\pi \le y-x \le \pi$, $\pi \le x+y \le 3\pi$

↳ Hallar transformación lineal : transformar región

$$\begin{cases} \mu = x+y \\ v = x \end{cases} \qquad \begin{cases} \mu - 2v = y-x \\ \mu - v = y \end{cases}$$

$-\pi \le y-x \le \pi \longrightarrow -\pi \le \mu - 2v \le \pi \to \underbrace{\frac{-\pi-\mu}{-2} \ge v \ge \frac{\pi-\mu}{-2}}_{\frac{\mu-\pi}{2} \le v \le \frac{\pi+\mu}{2}}$

$\pi \le x+y \le 3\pi \longrightarrow \pi \le \mu \le 3\pi$

↳ Hallar Jacobiano

$$|J(\mu,v)| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = |-1| = 1$$

y si pruebo $\mu = x-y$?

↳ Integrar

$$\iint_D (x-y)^2 \operatorname{sen}^2(x+y)\, dx\, dy = \int_{\pi}^{3\pi} \int_{\frac{\mu-\pi}{2}}^{\frac{\pi+\mu}{2}} (\mu-2v)^2 \operatorname{sen}^2(\mu)\, dv\, d\mu$$

↳ $\displaystyle\int_{\frac{\mu-\pi}{2}}^{\frac{\pi+\mu}{2}} (\mu-2v)^2 \operatorname{sen}^2(\mu)\, dv = \operatorname{sen}(\mu)\left(\mu^2 v - 2\mu v^2 + \frac{v^3}{3}\right)\Bigg|^{\frac{\pi+\mu}{2}}$

$$= \int (\mu^2 - 4\mu v + v^2) \operatorname{sen}^2(\mu)\, dv = \operatorname{sen}(\mu)\left[\int \mu^2 dv - 4\mu \int v\, dv + \int v^2 dv\right] = \operatorname{sen}(\mu)\left(\mu^2 v - 2\mu v^2 + \frac{v^3}{3}\right)$$

ⓒ (rehacer) $\iint_D (x-y)^2 \operatorname{sen}^2(x+y)\, dx\, dy$ D: $-\pi \le y-x \le \pi$, $\pi \le x+y \le 3\pi$

↳ Hallar transformación lineal ↳ transformar región

$$\begin{cases} \mu = x-y \\ v = x \end{cases} \to \begin{cases} v-\mu = y \\ 2v-\mu = x+y \end{cases}$$

$-\pi \le y-x \le \pi \longrightarrow -\pi \le \mu \le \pi$

$\pi \le x+y \le 3\pi \longrightarrow \pi \le 2v-\mu \le 3\pi \to \frac{\pi+\mu}{2} \le v \le \frac{3\pi+\mu}{2}$

↳ $|J(\mu,v)| = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = |1| = 1$

↳ calcular integral

$$\iint_D (x-y)^2 \operatorname{sen}^2(x+y)\, dx\, dy = \int_{-\pi}^{\pi} \int_{\frac{\pi+\mu}{2}}^{\frac{3\pi+\mu}{2}} \mu^2 \operatorname{sen}(2v-\mu)\, dv\, d\mu$$

$$\mu^2 \int_{\frac{\pi+\mu}{2}}^{\frac{3\pi+\mu}{2}} \operatorname{sen}^2\left(2\left[v - \frac{\mu}{2}\right]\right) dv$$

$$\mu^2 \int \operatorname{sen}^2\left(2\left[v-\frac{\mu}{2}\right]\right) dv = \mu^2 \int \left[2 \operatorname{sen}\left(v-\frac{\mu}{2}\right)\cos\left(v-\frac{\mu}{2}\right)\right]^2 dv$$

$$= \mu^2 \int 4 \operatorname{sen}^2\left(v-\frac{\mu}{2}\right)\cos^2\left(v-\frac{\mu}{2}\right) dv$$

4/6

ⓒ (rehacer)

ⓒ $\iint_D (x-y)^2 \operatorname{sen}^2(x+y)\, dx\, dy$ D: $-\pi \leq y-x \leq \pi$, $\pi \leq x+y \leq 3\pi$

↳ Hallar transformación lineal

$\begin{cases} \mu = x-y \longrightarrow -\mu = y-x \\ v = x+y \end{cases}$

$\begin{cases} v-\mu = 2y \longrightarrow y = \frac{v-\mu}{2} \\ \mu+v = 2x \longrightarrow x = \frac{v+\mu}{2} \end{cases}$

: transformar región

- $-\pi \leq y-x \leq \pi \longrightarrow -\pi \leq -\mu \leq \pi \longrightarrow \pi \geq \mu \geq -\pi$
- $\pi \leq v \leq 3\pi$

→ calcular jacobiana $|J(\mu,v)| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$

→ calcular Integral

$$\iint_D (x-y)^2 \operatorname{sen}(x+y)^2\, dx\, dy = \int_{\pi}^{3\pi} \int_{-\pi}^{\pi} \mu^2 \operatorname{sen}^2(v)\, \frac{1}{2}\, d\mu\, dv = \frac{\pi^3}{3} \int_{\pi}^{3\pi} \operatorname{sen}^2(v)\, dv = \frac{\pi^3}{3} \cdot (\pi) = \frac{\pi^4}{3}$$

- Calcular $\int_{-\pi}^{\pi} \mu^2 \operatorname{sen}^2(v)\, d\mu = \frac{\mu^3}{3} \operatorname{sen}^2(v) \Big|_{-\pi}^{\pi} = \left(\frac{\pi^3}{3} \operatorname{sen}^2(v)\right) - \left(\frac{-\pi^3}{3} \operatorname{sen}^2(v)\right) = \frac{2\pi^3}{3} \operatorname{sen}^2(v)$

$\int \mu^2 \operatorname{sen}^2(v)\, d\mu = \frac{\mu^3}{3} \operatorname{sen}^2(v)$

- Calcular $\int_{\pi}^{3\pi} \operatorname{sen}^2(v)\, dv = \frac{1}{2} v - \frac{\operatorname{sen}(2v)}{2} \Big|_{\pi}^{3\pi} = \left(\frac{3\pi}{2} - \frac{\operatorname{sen}(6\pi)}{2}\right) - \left(\frac{\pi}{2} - \frac{\operatorname{sen}(2\pi)}{2}\right) = \pi$

$\int \operatorname{sen}^2(v)\, dv = \int \frac{1-\cos(2v)}{2}\, dv = \frac{1}{2} \int 1 - \cos(2v)\, dv = \frac{1}{2} v - \frac{1}{2} \frac{1}{2} \operatorname{sen}(2v) = \frac{1}{2} v - \frac{\operatorname{sen}(2v)}{4}$

$1 - \cos^2(v)$

ⓓ $\iint_D (x+y)^3\, dx\, dy$ D: $1 \leq x+y \leq 4$, $-2 \leq x-2y \leq 1$

→ Hallar TL

$\begin{cases} \mu = x+y \\ v = x-2y \end{cases}$

$\begin{cases} x = \frac{2\mu+v}{3} \\ y = \frac{\mu-v}{3} \end{cases}$

→ transformar región D

$1 \leq x+y \leq 4 \longrightarrow 1 \leq \mu \leq 4$

$-2 \leq x-2y \leq 1 \longrightarrow -2 \leq v \leq 1$

$J(\mu,v) = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = \left|-\frac{1}{3}\right| = \frac{1}{3}$

$$\iint_D (x+y)^3\, dx\, dy = \int_1^4 \int_{-2}^{1} \mu^3 \frac{1}{3}\, dv\, d\mu = \frac{1}{3} \int_1^4 \mu^3 1 - (-2\mu^3)\, d\mu = \int_1^4 \mu^3\, d\mu = \frac{\mu^4}{4} \Big|_1^4 = \frac{4^4}{4} - \frac{1}{4} = \frac{255}{4}$$

* Ejercicio 10 : Resolver con coordenadas polares

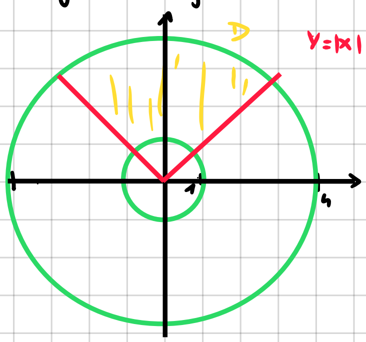
(a) Área círculo radio = 3 , centro origen

$$h(r,\theta) = (r\cos(\theta),\ r\,\text{sen}(\theta)) = (x(r,\theta),\ y(r,\theta)),\quad 0 \le r \le 3,\quad 0 \le \theta \le 2\pi$$

$$\iint_D dx\,dy = \int_0^{2\pi}\int_0^3 r.\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_0^3 = \int_0^{2\pi} \frac{3^2}{2}\,d\theta = \frac{9}{2}\theta\Big|_0^{2\pi} = 9\pi$$

(c) $\iint_D y^2\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 /\ 1 \le x^2+y^2 \le 16,\ \ y \ge |x|\}$

- gráfico región D



coordenadas polares

$$\begin{cases} x = r\cos(\theta) \\ y = r\,\text{sen}(\theta) \end{cases}$$

$$D^* = \left\{ (r\cos(\theta),\ r\,\text{sen}(\theta)) \,/\, 1 \le r \le 4,\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4} \right\}$$

$$\iint_D y^2\,dx\,dy = \iint_{D^*} (r\,\text{sen}(\theta))^2 . r .\ d\theta\,dr$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_1^4 (r\,\text{sen}(\theta))^2 r .\ dr\,d\theta = \frac{255}{4}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \text{sen}^2(\theta)\,d\theta = \frac{255}{4}\left(\frac{\pi}{4}+\frac{1}{2}\right) = \frac{255\pi}{16} + \frac{255}{8}$$

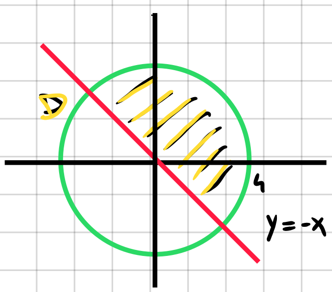
$$\int_1^4 (r\,\text{sen}(\theta))^2 r\,dr = \text{sen}^2(\theta)\int_1^4 r^3\,dr = \text{sen}^2(\theta)\,\frac{r^4}{4}\Big|_1^4 = \text{sen}^2(\theta)\left(\frac{255}{4}\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \text{sen}^2(\theta)\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\text{sen}(2\theta)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \left(\frac{3\pi}{8} - \frac{1}{4}\underbrace{\text{sen}\left(\frac{3\pi}{2}\right)}_{-1}\right) - \left(\frac{\pi}{8} - \frac{1}{4}\underbrace{\text{sen}\left(\frac{\pi}{2}\right)}_{1}\right) = \frac{\pi}{4} + \frac{1}{4} + \frac{1}{4} = \frac{\pi}{4} + \frac{1}{2}$$

$$\int \text{sen}^2(\theta)\,d\theta = \int \frac{1-\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\int 1-\cos(2\theta)\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\text{sen}(2\theta)$$

(b) $\iint_D 2x\,dx\,dy$, $D = \{(x,y) /\ x^2+y^2 \le 16 .\ x+y \ge 0\}$

gráfico de D



coordenadas polares $\begin{cases} x = r\cos(\theta) \\ y = r\,\text{sen}(\theta) \end{cases}$, $r \ge 0$

→ transformar D a coord polares

$$x^2+y^2 \le 16 \to r^2(\cos^2+\text{sen}^2) \le 16 \to r \le 4 \to 0 \le r \le 4$$

$$x+y \ge 0 \to r\cos(\theta) + r\,\text{sen}(\theta) \ge 0 \to r(\cos(\theta)+\text{sen}(\theta)) \ge 0$$

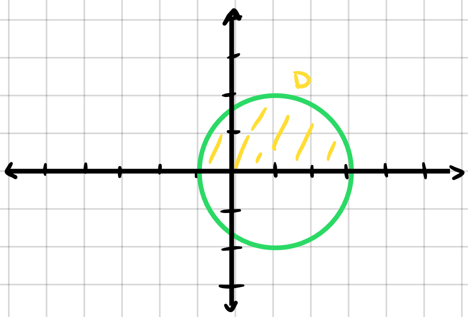
$$\cos(\theta) + \text{sen}(\theta) \ge 0 \to 1 \ge \frac{-\text{sen}(\theta)}{\cos(\theta)}$$

$$1 \ge -\,\text{tg}(\theta) \to \text{tg}(\theta) \ge -1 \qquad -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$$

$$\iint_D 2x\,dx\,dy = 2\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^4 r\cos\theta\,r\,dr\,d\theta = 2\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{r^3}{3}\cos\theta\Big|_0^4 d\theta = \frac{128}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos(\theta)\,d\theta = \frac{128}{3}\left[\text{sen}(\theta)\Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\right] = \underbrace{\frac{128}{3}\left[\underbrace{\text{sen}\left(\frac{3\pi}{4}\right) - \text{sen}\left(-\frac{\pi}{4}\right)}_{\sqrt{2}}\right]}_{\frac{128\sqrt{2}}{3}}$$

d) $\iint_D xy\, dx\, dy \qquad D=\{(x,y)\in\mathbb{R}^2 / \underbrace{(x-1)^2+y^2\le 4}_{\substack{\text{centro}=(1,0)\\ r=2}},\ y\ge 0\}$

Gráfico D



• coord polares

$$\begin{cases} x-1 = r.\cos(\theta) \rightarrow x = r\cos(\theta)+1 \\ y = r.\mathrm{sen}(\theta) \end{cases}, \text{ donde } r\ge 0$$

• transformar D a coord polares

$\rightarrow (x-1)^2+y^2\le 4 \rightarrow r^2(\cos^2\theta+\mathrm{sen}^2\theta)\le 4 \rightarrow \quad 0\le r\le 2$

$\rightarrow \quad y\ge 0 \rightarrow r\,\mathrm{sen}\theta \ge 0 \rightarrow \quad 0\le\theta\le\pi$

$$\iint_D xy\, dx\,dy = \int_0^{\pi}\int_0^{2}[r\cos(\theta)+1][r\,\mathrm{sen}(\theta)]\cdot r\, dr\, d\theta = \frac{16}{3}$$

$$\int_0^2 (r\cos t+1)(r\,\mathrm{sen}(\theta))\, r\, dr = \int_0^2 r^3\cos(\theta)\mathrm{sen}(\theta) + r^2\mathrm{sen}(\theta)\, dr = \int_0^{\pi} 2\,\mathrm{sen}(2\theta)+\frac{8}{3}\mathrm{sen}(\theta)\, d\theta = \frac{16}{3}$$

$$= \frac{r^4}{4}\cos(\theta)\mathrm{sen}(\theta) + \frac{r^3}{3}\mathrm{sen}(\theta)\Big|_0^2 = 2\,\mathrm{sen}(2\theta)+\frac{8}{3}\mathrm{sen}(\theta)$$

$$\int_0^{\pi} 2\,\mathrm{sen}(2\theta)+\frac{8}{3}\mathrm{sen}(\theta)\, d\theta = -\cos(2\theta)-\frac{8}{3}\cos(\theta)\Big|_0^{\pi} = \left(-1-\frac{8}{3}(-1)\right)-\left(-1-\frac{8}{3}\right) = -1+\frac{8}{3}+1+\frac{8}{3} = \frac{16}{3}$$

---

e) $\iint_D e^{x^2+y^2}\, dx\,dy \qquad$ D. circulo radio R con centro $(0,0)$ $\rightarrow D=\{(x,y)/ x^2+y^2\le r^2\}$

→ coordenadas polares

$$\begin{cases} x = r\cos(\theta) \\ y = r\,\mathrm{sen}(\theta) \end{cases}, \quad r\in\mathbb{R}\ge 0, \quad 0\le\theta\le 2\pi$$

↳ $x^2+y^2 = r^2(\cos^2(\theta)+\mathrm{sen}^2(\theta)) = r^2$

↳ reescribir integral

$$\iint_D e^{x^2+y^2}dx\,dy = \iint_{D^*} e^{r^2}\cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{R} e^{r^2}\cdot r\, dr\, d\theta = \int_0^{2\pi}\left(\frac{e^{r^2}}{2}\Big|_0^R\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{e^{R^2}}{2}-\frac{1}{2}\right)d\theta = \frac{1}{2}\left(e^{R^2}\theta - \theta\right)$$

$$= \frac{1}{2}\left(e^{R^2}2\pi - 2\pi\right) = \pi\left(e^{R^2}-1\right)$$

(f) $\iint_D \frac{x+y}{x^2}\,dx\,dy$ $\quad D: \; 0 \le y \le x \quad x+y \le 2$

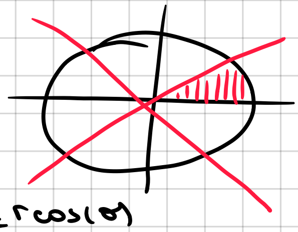
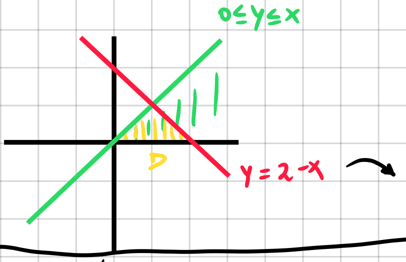


gráfico D



$D = \{(x,y) \,/\; y < x \le 2-y, \; 0 \le y \le 1\}$

$$\int_0^1 \int_y^{2-y} \frac{x+y}{x^2}\,dx\,dy$$

transformar D

$$\begin{cases} x = r\cos(\theta) \\ y = r\,\mathrm{sen}(\theta) \end{cases} \rightarrow \quad 0 \le y \le x \rightarrow \quad 0 \le r\,\mathrm{sen}(\theta) \le r\cos(\theta)$$

$\rightarrow x+y \le 2$ $\qquad \mathrm{sen}(\theta) \le \cos(\theta) \rightarrow tg(\theta) \le 1$

$r\cos(\theta) + r\,\mathrm{sen}(\theta) \le 2$ $\qquad \boxed{0 \le \theta \le \frac{\pi}{4}}$

$r(\cos(\theta) + \mathrm{sen}(\theta)) \le 2 \longrightarrow 0 \le r \le \frac{2}{(\cos(\theta) + \mathrm{sen}(\theta))}$

$J(r,\theta) = r$

$$\iint_D \frac{x+y}{x^2}\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos(\theta)+\mathrm{sen}(\theta)}} \frac{r\cos(\theta) + r\,\mathrm{sen}\,\theta}{r^2\cos^2(\theta)}\, r\,dr\,d\theta$$

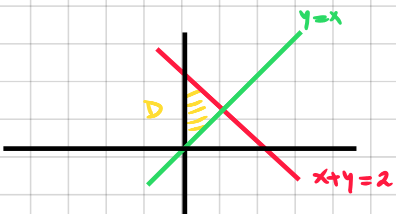
$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos(\theta)+\mathrm{sen}(\theta)}} \frac{r\cos(\theta) + r\,\mathrm{sen}\,\theta}{r^2\cos^2(\theta)}\, r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \frac{2}{\cos^2(\theta)}\,d\theta = 2\int \frac{1}{\cos^2(\theta)}\,d\theta = 2tg(\theta)\Big|_0^{\frac{\pi}{4}}$$

$$\int r\frac{(\cos(\theta) + r\,\mathrm{sen}(\theta))\,r}{r^2\cos^2(\theta)}\,dr = \int \frac{\cos(\theta) + \mathrm{sen}\,\theta\; r}{r\cos^2(\theta)}\,dr = \frac{\cos + \mathrm{sen}}{\cos^2(\theta)} \int 1\,dr = \frac{\cos + \mathrm{sen}}{\cos^2(\theta)} \cdot r \Big|_0^{\frac{2}{\cos+\mathrm{sen}}}$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{2}{\cos(\theta)+\mathrm{sen}(\theta)}} \frac{r\cos(\theta) + r\,\mathrm{sen}\,\theta}{r^2\cos^2(\theta)}\, r\,dr\,d\theta = 2tg\,\theta\Big|_0^{\frac{\pi}{4}} = 2tg\left(\frac{\pi}{4}\right) - 2tg(0) = 2$$

(g) $\iint_D \frac{1}{(x+y)}\,dx\,dy$ $\quad D = \; x \ge 0, \; x+y \le 2, \; y \ge x$

Gráfico D



$$\begin{cases} x = r\cos(\theta) \\ y = r\,\mathrm{sen}(\theta) \end{cases}$$

transform región D

$x \ge 0 \longrightarrow r\cos(\theta) \ge 0$ $\quad r \ge 0$, $\; \cos(\theta) \ge 0$ $\rightarrow \theta \le \frac{\pi}{2}$

$x+y \le 2 \rightarrow r\cos(\theta) + r\,\mathrm{sen}(\theta) \le 2 \rightarrow r \le \frac{2}{\cos(\theta)+\mathrm{sen}(\theta)}$

$y \ge x \longrightarrow r\,\mathrm{sen}(\theta) \ge r\cos(\theta)$

$tg(\theta) \ge 1 \longrightarrow \theta \ge \frac{\pi}{4}$

$J(r,\theta) = r$

$$\iint_D \frac{1}{(x+y)}\,dx\,dy = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{2}{\cos(\theta)+\mathrm{sen}(\theta)}} \frac{1}{r\cos(\theta) + r\,\mathrm{sen}(\theta)} \cdot r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{r}{\cos(\theta) + \mathrm{sen}(\theta)} \Big|_0^{\frac{2}{\cos(\theta)+\mathrm{sen}(\theta)}} \right) d\theta$$

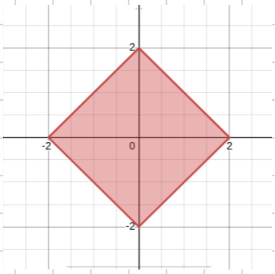
$$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{\cos(\theta) + \mathrm{sen}\,\theta}\right)^2 d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\cos^2(\theta) + \mathrm{sen}(2\theta) + \underline{\mathrm{sen}^2\theta}}\,d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{1 + \mathrm{sen}(2\theta)}\,d\theta$$

|5/6/2025|

* Ejercicio 11 Resolver con cambio de coord propuesto

ⓐ área (D), $D = \{(x,y) \in \mathbb{R}^2 \;/\; |x| + |y| \le 2\}$, $(x,y) = \left(\frac{\mu+v}{2}, \frac{\mu-v}{2}\right)$

→ Gráfico D



Reescribir D

$|x| + |y| \le 2$

$x + y \le 2$ si $x \ge 0, y \ge 0$

$-x + y \le 2$ si $x \le 0, y \ge 0$

$x - y \le 2$ si $x \ge 0, y \le 0$

$-x - y \le 2$ si $x \le 0, y \le 0$

Transformar D a D*

$x + y \le 2 \longrightarrow \frac{\mu+v+\mu-v}{2} \le 2 \to \mu \le 2$

$x - y \le 2 \longrightarrow \frac{\mu+v-\mu+v}{2} \le 2 \to v \le 2$

$-x + y \le 2 \longrightarrow -\frac{\mu-v}{2} + \frac{\mu-v}{2} \le 2 \to v \ge -2$

$-x - y \le 2 \longrightarrow \frac{-\mu-v-\mu+v}{2} \le 2 \to -\mu \le 2 \to \mu \ge -2$

$-2 \le \mu \le 2 \;,\; -2 \le v \le 2$ → μ y v son rectos

$|J(\mu,v)| = \left|\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}\right| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$

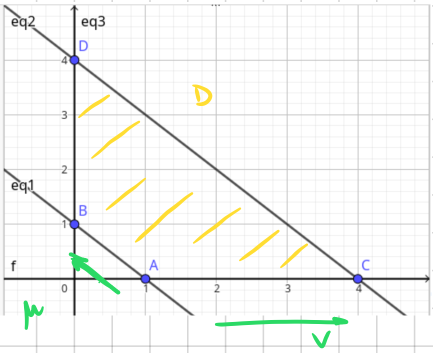
$$\text{área}(D) = \iint_D 1 \, dx\,dy = \iint_{D^*} 1 \, |J(\mu,v)| \, d\mu\,dv = \frac{1}{2} \cdot 4 \cdot 4 = 8$$

$$= \int_{-2}^{2}\int_{-2}^{2} \left[\frac{\mu+v}{2} + \frac{\mu-v}{2}\right] \cdot \frac{1}{2} \, dv\,d\mu =$$

$$\Rightarrow \int_{-2}^{2} \frac{\mu+v+\mu-v}{2} \, dv = \int_{-2}^{2} \mu \, dv$$

(b) $\iint_D e^{x+y}\,dx\,dy$, $D = 1 \le x+y \le 4$, $x \ge 0$, $y \ge 0$

$\Rightarrow$ gráfico D



transformación lineal

$$\begin{cases} \mu = x+y \\ v = x \end{cases} \longrightarrow \begin{cases} x = v \\ y = \mu - v \end{cases}$$

Jacobiano

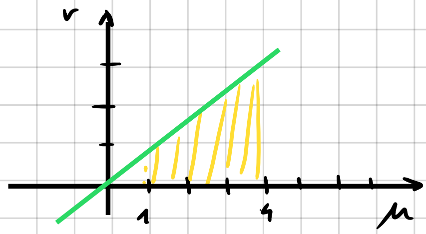
$$|J(\mu,v)| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = |-1| = 1$$

Hallar lim de integración

$1 \le x+y \le 4 \longrightarrow 1 \le \mu \le 4$

$x \ge 0 \longrightarrow v \ge 0$

$y \ge 0 \longrightarrow \mu - v \ge 0 \longrightarrow v \le \mu$



graf transf lineal

$$\text{área}(D) = \iint_D e^{x+y}\,dx\,dy = \iint_{D^*} e^{\mu}\,|J(\mu,v)|\,d\mu\,dv$$

$$\int_1^4 \int_0^{\mu} e^{\mu}\,dv\,d\mu = \int_1^4 \left(e^{\mu} v \Big|_0^{\mu}\right) d\mu = \int_1^4 \mu e^{\mu}\,d\mu = \mu e^{\mu} - e^{\mu}\Big|_1^4 = [4e^4 - e^4] - [\underbrace{e-e}_{0}] = 3e^4$$

$a = \mu \to a' = 1$
$b' = e^{\mu} \to b = e^{\mu}$

(c) area (D) $\quad D = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, $a,b \in \mathbb{R}^+ \longrightarrow D$ es una elipse

coordenadas polares

$$\begin{cases} x = a\,r\cos(\theta) \\ y = b\,r\,\mathrm{sen}(\theta) \end{cases}$$

$$|J(r,\theta)| = \begin{vmatrix} a\cos(\theta) & -ar\,\mathrm{sen}(\theta) \\ b\,\mathrm{sen}(\theta) & br\cos(\theta) \end{vmatrix} = abr\cos^2(\theta) - (-abr\,\mathrm{sen}^2) = abr(\cos^2 + \mathrm{sen}^2)$$

Hallar limites de integración

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \longrightarrow \frac{[ar\cos(\theta)]^2}{a^2} + \frac{[br\,\mathrm{sen}(\theta)]^2}{b^2} \le 1 \longrightarrow r^2\cos^2(\theta) + r^2\mathrm{sen}^2(\theta) \le 1$$

$$r^2(\cos^2\theta + \mathrm{sen}^2\theta) \le 1 \longrightarrow \boxed{r \le 1}$$

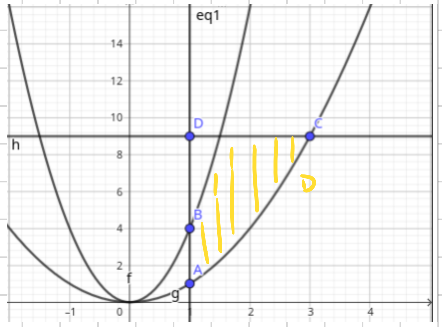
$0 \le r \le 1$

$0 \le \theta \le 2\pi \longrightarrow$ toda la elipse

$$\text{area}(D) = \int_0^{2\pi} \int_0^1 abr\,.\,dr\,d\theta = \int_0^{2\pi} \left(\frac{abr^2}{2}\Big|_0^1\right) d\theta = \int_0^{2\pi} \frac{ab}{2}\,d\theta = ab\pi$$

(d) $\iint_D x^{-1}\,dx\,dy \qquad D = \quad x^2 \le y \le 4x^2, \quad x \ge 1, \quad y \le 9$

→ región D



→ transformación

$$\begin{cases} x = v/\mu \\ y = v^2/\mu \end{cases}$$

→ Jacobiano

$$J(\mu,v) = \begin{vmatrix} -\frac{v}{\mu^2} & \frac{1}{\mu} \\ -\frac{v^2}{\mu^2} & \frac{2v}{\mu} \end{vmatrix} = -\frac{2v^2}{\mu^3} - \left(-\frac{v^2}{\mu^3}\right) = \frac{v^2}{\mu^3}$$

→ Hallar límites de integración

$$\mu^2 \ge \mu \ge \frac{\mu^2}{4}$$

$$x^2 \le y \le 4x^2 \to \frac{v^2}{\mu^2} \le \frac{v^2}{\mu} \le \frac{4v^2}{\mu^2} \to \frac{1}{\mu^2} \le \frac{1}{\mu} \le \frac{4}{\mu^2}$$

$$x \ge 1 \to \frac{v}{\mu} \ge 1 \to v \ge \mu$$

$$y \le 9 \to \frac{v^2}{\mu} \le 9 \to |v| \le 3\sqrt{\mu}$$

$$\mu \le v \le 3\sqrt{\mu}$$

$$\mu \ge 1 \ \wedge\ \frac{\mu}{4} \le 1$$
$$1 \ge \mu \le 4$$

$$\iint_D x^{-1}\,dx\,dy = \int_1^4 \int_\mu^{3\sqrt{\mu}} \frac{\mu}{v} \cdot \frac{v^2}{\mu^3}\,dv\,d\mu = \int_1^4 \frac{9}{2}\frac{1}{\mu} - \frac{1}{2} = \frac{9}{2}\ln(\mu) - \frac{1}{2}\mu \Big|_1^4 = \left(\frac{9}{2}\ln(4) - 2\right) - \left(-\frac{1}{2}\right)$$

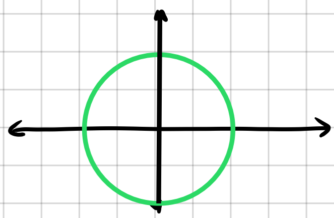
$$= \frac{9}{2}\ln(4) - \frac{3}{2}$$

$$\int_\mu^{3\sqrt{\mu}} \frac{v}{\mu^2}\,dv = \frac{v^2}{2\mu^2}\Big|_\mu^{3\sqrt{\mu}} = \frac{9\mu}{2\mu^2} - \frac{\mu^2}{2\mu^2} = \frac{9}{2\mu} - \frac{1}{2} =$$

* Ejercicio 12 $\quad \int_0^4 f(t)\,dt = 1 \quad$ f contina

↳ Hallar $\iint_D f(x^2+y^2)\,dx\,dy$ siendo $D: x^2+y^2 \le 4$

→ D es círculo radio 2



coord polares $\quad h(r,\theta)$

$$\begin{cases} x = r\cos(\theta) \\ y = r\,\mathrm{sen}(\theta) \end{cases} \qquad |J(r,\theta)| = r$$

$$x^2+y^2 = [r\cos(\theta)]^2 + [r\,\mathrm{sen}(\theta)]^2 = r^2(\cos^2\theta + \mathrm{sen}^2\theta) = r^2$$

$$0 \le r^2 \le 4$$

$$\iint_D f(x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^4 f(r^2)\,r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}\,d\theta = \frac{1}{2}\theta\Big|_0^{2\pi} = \pi$$

$$\int_0^4 f(r^2)\,r\,dr \quad \frac{1}{2}\int_0^4 f(t)\,dt = \frac{1}{2}$$

$t = r^2$

$dt = 2r\,dr$

$\frac{dt}{2} = r\,dr$