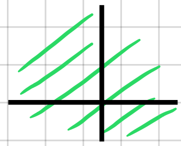
3/4/2025 Func varias variables ; Continuidad ; Derivabilidad ; Diferenciabilidad . Superficies

⇒ CONJUNTOS DE NIVEL

* Ejercicio 1 → Det dom f , graf dom f analizar tipo de conjunto, Conj nivel f y gráf N_k

ⓐ $f(x,y) = 3\left(1-\frac{x}{2}-\frac{y}{2}\right)$

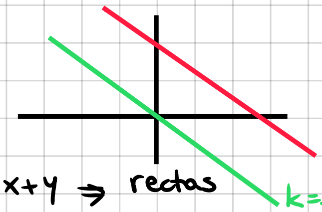


• $\text{Dom } f = \{(x,y)\in\mathbb{R}^2\}$ → conjunto abierto no acotado (todo $\mathbb{R}^2$)

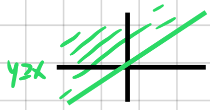
• Conjunto de nivel k

$$N_k = \{(x,y)\in\mathbb{R} \;/\; 3\left(1-\tfrac{x+y}{2}\right) = k ,\ k\in\mathbb{R}\}$$

$$3\left(1-\tfrac{x+y}{2}\right) = k \Rightarrow 1-\frac{k}{3} = \frac{x+y}{2} \Rightarrow 2-\frac{2k}{3} = x+y \Rightarrow \text{rectas}$$



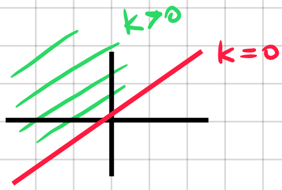
ⓑ $f(x,y) = \sqrt{y-x}$



• Hallar dom $\quad y-x\geq 0 \Rightarrow y\geq x \qquad \text{Dom } f = \{(x,y)\in\mathbb{R}^2 / y\geq x\}$ → cerrado , no acotado

$\nexists\, B(r,A)$ con $r\in\mathbb{R}$

• Hallar conj Nivel N_k



$$N_k = \{(x,y)\in D \;/\; \sqrt{y-x} = k\}$$

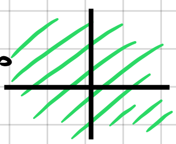
$\sqrt{y-x} = k$ → $k=0 \Rightarrow \sqrt{y-x} = 0 \Rightarrow \boxed{y=x}$ recta identidad

→ $k>0 \Rightarrow \underbrace{\sqrt{y-x}}_{\geq 0} = k \Rightarrow \boxed{y = k^2+x}$ rectas pendiente $m=1$ ord origen $b=k^2$

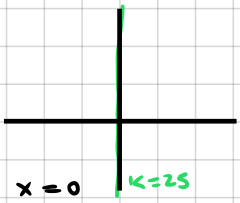
→ $k<0 \Rightarrow \nexists\ \sqrt{y-x} = k \quad \phi$

ⓒ $f(x,y) = 25-x^2$

conj abierto no acotado



• $\text{Dom }(f) = \{(x,y)\in\mathbb{R}^2\}$



• $\text{Conj Nivel}_k = \{(x,y)\in\mathbb{R}^2 / 25-x^2 = k\}$

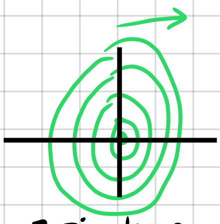
si $k=25 \to x=0$

si $k<25 \to |x| = \sqrt{25-k}$ → rectas verticales $(|x|, y)$ en $\mathbb{R}^2$

$k>25 \to \phi$

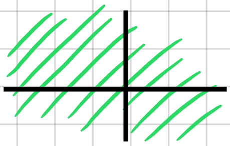
$$25-x^2 = k \Rightarrow \underbrace{x^2}_{\geq 0} = \underbrace{25-k}_{\geq 0} \qquad \boxed{\begin{matrix}25-k\geq 0\\ k\leq 25\end{matrix}}$$

$$N_k = \{(x,y)\in\mathbb{R}^2 / \ 25-x^2=k ,\ k\leq 25\}$$



ⓓ $f(x,y) = 9x^2+4y^2$

• $\text{Dom }(f): \{(x,y)\in\mathbb{R}^2\}$



conj abierto no acotado

• Conjuntos de nivel $N_k = \{(x,y)\in\mathbb{R}^2 / \ 9x^2+4y^2 = k\}$

si $k>0$ → elipse

si $k=0 \to \phi$

si $k<0 \to \phi$

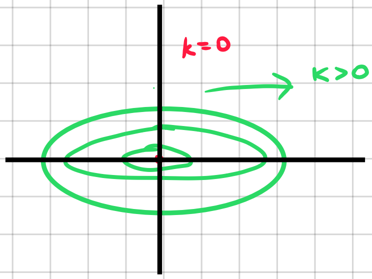
$$9x^2+4y^2 = k$$

$$\frac{9x^2}{k} + \frac{4y^2}{k} = 1 \Rightarrow \underbrace{\frac{x^2}{\left(\frac{k}{9}\right)} + \frac{y^2}{\left(\frac{k}{4}\right)}}_{\boxed{k>0}} = 1 \to \text{elipses} \quad \left.\begin{matrix} a=\sqrt{\frac{k}{9}} \\ b=\sqrt{\frac{k}{4}}\end{matrix}\right\} \to \boxed{k\geq 0}$$

ⓔ $f(x,y) = \sqrt{x^2+2y^2}$

• $\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2\}$

$\underbrace{x^2+2y^2}_{\text{siempre positivo}} \geq 0$



• Conjunt Nivel $N_k = \{(x,y) \in \mathbb{R}^2 / \sqrt{x^2+2y^2} = k\}$

$\sqrt{x^2+2y^2} = k \to |x^2+2y^2| = k^2 \to \underbrace{x^2+2y^2 = k^2}_{\text{elipse}}$

$\to \frac{x^2}{(k^2)} + \frac{y^2}{\left(\frac{k^2}{2}\right)} = 1$, donde $k>0$

$\begin{cases} \text{si } k>0 \longrightarrow \frac{x^2}{k^2} + \frac{y^2}{\frac{k^2}{2}} = 1 \to \text{elipses} \\ \text{si } k=0 \longrightarrow \sqrt{x^2+2y^2} = k^2 \to (0,0) \\ \text{si } k<0 \longrightarrow \phi \end{cases}$

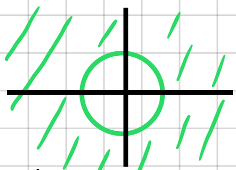
ⓕ $f(x,y) = (x^2+y^2-16)^{-\frac{1}{2}} = \frac{1}{\sqrt{(x^2+y^2-16)}}$

$\text{Dom}(f) \to \{(x,y) \in \mathbb{R}^2 / x^2+y^2 > 16\}$

$\downarrow$

$x^2+y^2-16 > 0$

$x^2+y^2 > 16$



conjunto abierto, no acotado

• $N_k = \{(x,y) \in D / \frac{1}{\sqrt{x^2+y^2-16}} = k\}$

$\frac{1}{\sqrt{x^2+y^2-16}} = k$

$\underbrace{k}_{\neq 0} \underbrace{\sqrt{x^2+y^2-16}}_{>0 \text{ En Dom}(f)} = 1 \longrightarrow x^2+y^2-16 = \frac{1}{k}$

$\longrightarrow \underbrace{x^2+y^2}_{>0} = \underbrace{\frac{1}{k}+16}_{>0}$

• Niveles disponibles (k)

si $k=0 \longrightarrow \phi$

si $k > \frac{-1}{16} \wedge k \neq 0 \to$ circunferencias

si $k \leq \frac{-1}{16} \longrightarrow \phi$

Hallar k para $\frac{1}{k}+16 \geq 0$

$\frac{1}{k}+16>0 \qquad -16k<1$

$\frac{1}{k} > -16 \qquad k > \frac{1}{-16}$

ⓖ $f(x,y) = e^{-x^2-y^2} = e^{-(x^2+y^2)} = \frac{1}{e^{x^2+y^2}}$

$\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2\}$

$\downarrow$

$\underbrace{e^{\overbrace{x^2+y^2}^{\geq 0}}}_{\geq 0}$

función exponencial
nunca negativo

• Hallar conjunto Nivel : $\{(x,y) \in \mathbb{R}^2 / e^{-x^2-y^2} = k\}$

$\frac{1}{e^{x^2+y^2}} = k \to e^{x^2+y^2} = \frac{1}{k}$

$\underbrace{x^2+y^2 = \ln\left(\frac{1}{k}\right)}_{\text{circunferencias}}, \quad \underbrace{k \neq 0 \wedge \frac{1}{k} > 0}_{k>0}$

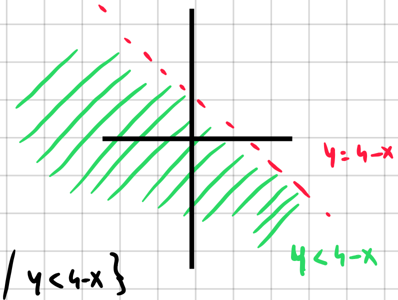
$\begin{cases} \text{si } k=1 \longrightarrow x^2+y^2=0 \longrightarrow (0,0) \\ \text{si } k>0 \wedge k\neq 1 \longrightarrow x^2+y^2 = \ln\left(\frac{1}{k}\right) \longrightarrow \text{circunferencias radio } \sqrt{\ln\left(\frac{1}{k}\right)} \\ \text{si } k<0 \longrightarrow \phi \end{cases}$

(h) $f(x,y,z) = \ln(4-x-y)$

• Hallar dominio

$4-x-y > 0$

$4-x > y$



$\text{Dom}(f) = \{(x,y,z) \in \mathbb{R}^3 / y < 4-x\}$

↪ conj abierto ⟶ no incluye frontera $4-x=y$

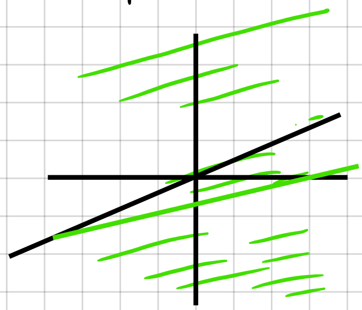
↪ no acotado

• Hallar conjunto nivel $N_k = \{(x,y,z) \in \mathbb{R}^3 / \ln(4-x-y) = k\}$

$\ln(4-x-y) = k$

$4-x-y = e^k$

$\underbrace{4-e^k-x = y}_{\text{plano en } \mathbb{R}^3}$



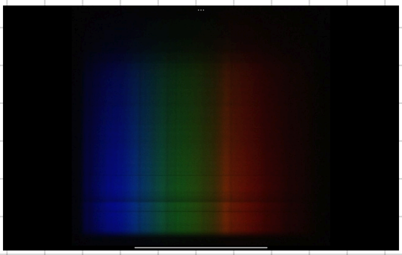
(I) $f(x,y,z) = x+y+2z$

$\text{Dom}(f) = \mathbb{R}^3$

• $N_k = \{(x,y,z) \in \mathbb{R}^3 / \underbrace{x+y+2z=k}_{\text{plano}}\}$

(J) $f(x,y,z) = e^{x^2+y^2-z^2}$

$\text{Dom}(f) = \{(x,y,z) \in \mathbb{R}^3\}$
función exponencial continua en $\mathbb{R}^3$



• $N_k = \{(x,y,z) \in \mathbb{R}^3 / e^{x^2+y^2-z} = k\}$

$e^{x^2+y^2+2z} = k$

$x^2+y^2+2z = \underbrace{\ln(k)}_{\to \in \mathbb{R}}$ si $k \le 0 \longrightarrow N_k = \emptyset$

$\dfrac{x^2+y^2-\ln(k)}{-2} = z \Rightarrow \underbrace{z = \dfrac{-x^2}{2} - \dfrac{y^2}{2} + \ln(k)}_{(?)}$

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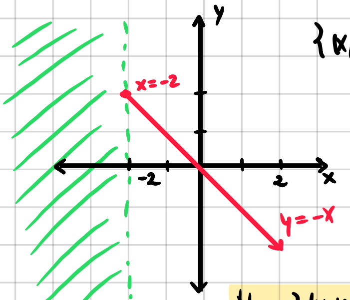
* Ejercicio 2 = Graf, si quedan def, N_0 y N_4

(a) $f(x,y) = \begin{cases} x+y & \text{si } x \ge -2 \\ 0 & \text{si } x < -2 \end{cases}$

• Hallar Dominio $\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2\}$

• Hallar $N_0 \to \{(x,y) \in D / f(x,y) = 0\}$

$\{(x,y) \in D / x < -2\}$



¿Qué pasa si $x \ge -2$ y $f(x,y) = 0$

$x+y=0 \Rightarrow y=-x$

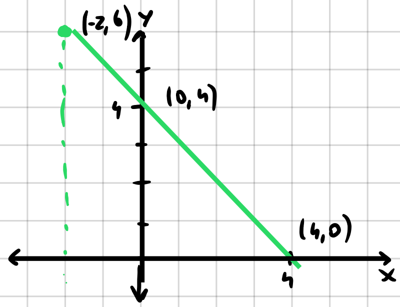
$N_0 = \{(x,y) \in \mathbb{R}^2 / x < -2\} \cup \{(x,y) \in D / y=-x, x \ge -2\}$

• Hallar $N_4 = \{(x,y) \in D / f(x,y) = 4\}$

si $x \ge -2$

$x+y=4 \Rightarrow y = 4-x$ recta

$N_4 = \{(x,y) \in D / y = 4-x \wedge x \ge -2\}$



* ⓑ $f(x,y) = \text{sen}(y-x) \rightarrow$ función periódica $\rightarrow f(x,y) \in [0,1]$

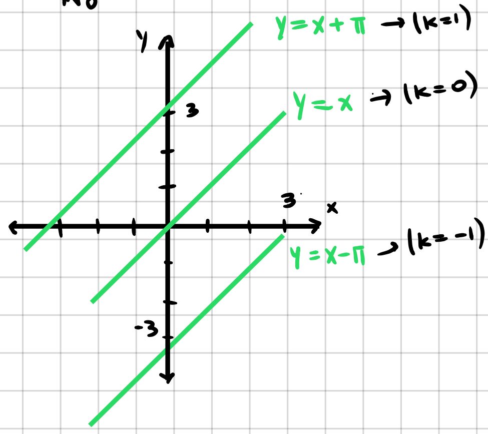
- Hallar dominio $\text{Dom} f = \{(x,y) \in \mathbb{R}^2\}$
- Hallar conjunto Nivel 0

$f(x,y) = \text{sen}(y-x) = 0$

$y - x = \text{arcsen}(0) = \underbrace{\pi k,\ k \in \mathbb{Z}}$ ↗ función periódica

$y = x + \pi k \Rightarrow$ rectas en $\mathbb{R}^2$

- Gráfico N_0



$N_4 = \phi$

- Hallar $N_4 \rightarrow \nexists f(x,y) = 4 \rightarrow f(x,y)$ oscila entre $[0,1]$

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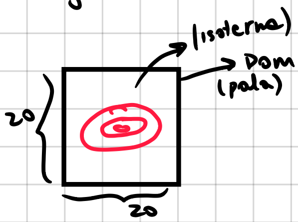
* $D = \{(x,y) \in \mathbb{R}^2 : x \in [-10,10],\ y \in [-10,10]\} = [-10,10] \times [-10,10] \longrightarrow$ placa metálica

$T(x,y) = 64 - 4x^2 - 8y^2 \longrightarrow$ temperatura de punto $(x,y) \in D$

Dibujar "isotermas" $\longrightarrow$ Conj de puntos con misma temperatura k

- Figura de análisis



- Hallar conjunto de nivel N_k (en general) ↗ Hallar isotermas de Temperatura k

$T(x,y) = 64 - 4x^2 - 8y^2 = k$

$64 - k = 4x^2 + 8y^2 \Rightarrow$ elipses

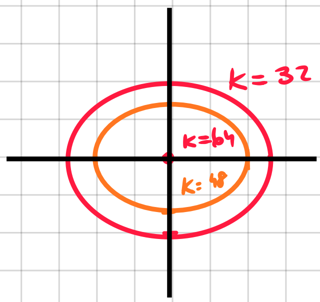
$1 = \dfrac{4x^2 + 8y^2}{64-k} \Rightarrow 1 = \dfrac{x^2}{\left(\frac{64-k}{4}\right)} + \dfrac{y^2}{\left(\frac{64-k}{8}\right)}$ ↗ $64 - k > 0$ $\boxed{k < 64}$

- Gráfico

$k = 32 \Rightarrow 1 = \dfrac{x^2}{8} + \dfrac{y^2}{4}$ $\begin{cases} a = \sqrt{8} \cong 2{,}82 \\ b = 2 \end{cases}$

$k = 48 \Rightarrow 1 = \dfrac{x^2}{4} + \dfrac{y^2}{2}$ $\begin{cases} a = 2 \\ b = \sqrt{2} \cong 1{,}41 \end{cases}$

$k = 64 \Rightarrow 0 = 4x^2 + 8y^2 \longrightarrow (0,0)$



$k = 16 \Rightarrow 1 = \dfrac{x^2}{12} + \dfrac{y^2}{6}$

$k = 8 \Rightarrow 1 = \dfrac{x^2}{14} + \dfrac{y^2}{7}$

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* Ejercicio 4 $U(x,y)$ esp potencial electrostático $(x,y) \in D \subset \mathbb{R}^2$

$U(x,y) = k/\sqrt{x^2+y^2}$ $(x,y) \in \mathbb{R}^2 - \{\bar{0}\}$ siendo $k \in \mathbb{R}_{>0}$

- Hallar líneas equipotenciales (E)

$U(x,y) = \dfrac{k}{\underbrace{\sqrt{x^2+y^2}}_{>0\ \forall (x,y) \in \mathbb{R}^2}} = E \rightarrow \left(\dfrac{k}{E}\right)^2 = x^2 + y^2$ ↗ circunferencias en $\mathbb{R}^2$

de radio $r = \dfrac{k}{E}$, y $E \neq 0$

no existe equipotencial 0

LÍMITES Y CONTINUIDAD DE CAMPOS

* **Ejercicios** : Determinar existencia del lím, fundamentar rta

(a) $\lim\limits_{(x,y)\to(1,1)} xy - y^2 = 0 \Rightarrow \exists \lim\limits_{(x,y)\to(1,1)} (xy - y^2) = 0$ (existe lim y es 0)

(b) $\lim\limits_{(x,y)\to(0,4)} xy^{-\frac{1}{2}} = \lim\limits_{(x,y)\to(0,4)} \frac{1}{\sqrt{xy}} = \infty$ no existe lim

(c) $\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0 \Rightarrow \exists \lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$ existe lim y es 0

(d) $\lim\limits_{(x,y)\to(0,0)} \frac{x}{x+y} \Rightarrow \text{ind } \frac{0}{0}$ probar acercamiento (0,0) por curvas

- familia de rectas (0,0) → $y = mx$

$f(x, mx) = \frac{x}{x+mx} = \frac{x}{x(1+m)}$

$\Rightarrow \lim\limits_{x\to 0} \frac{x}{x(1+m)} = \lim\limits_{x\to 0} \frac{1}{1+m} = \frac{1}{1+m}$

→ si $m = 1$ → $\lim\limits_{x\to 0} f(x, xm) = \frac{1}{2}$

→ si $m = 2$ → $\lim\limits_{x\to 0} f(x, mx) = \frac{1}{3}$

→ si limite existe es único → $\nexists \lim\limits_{(x,y)\to(0,0)} \frac{x}{x+y}$

(e) $\lim\limits_{(x,y)\to(0,2)} \frac{x^2(y-2)^2}{x^2+(y-2)^2} = \text{Ind } \frac{0}{0} \Rightarrow \lim\limits_{(x,y)\to(0,2)} \frac{x^2 (y-2)^2}{x^2+(y-2)^2} = 0$

acotado

⇒ Probar acotada

$x^2 \leq x^2 + \underbrace{(y-2)^2}_{\geq 0} \rightarrow \frac{x^2}{x^2+(y-2)^2} \leq 1 \rightarrow \underbrace{\left| \frac{x^2}{x^2+(y-2)^2} \right|}_{>0 \ \forall (x,y)\in\mathbb{R}^2} \leq \frac{1}{M}$

(f) $\lim\limits_{x\to 0} \left(\frac{x}{|x|}, \sqrt{x} \right)$ $f_1 = \begin{cases} \frac{x}{x} & \text{si } x > 0 \\ \frac{-x}{x} & \text{si } x < 0 \end{cases}$

- En f_1 $\lim\limits_{x\to 0} f_1(x) = \begin{cases} \lim\limits_{x\to 0^+} \frac{x}{x} = 1 \\ \lim\limits_{x\to 0^-} \frac{-x}{x} = -1 \end{cases}$

como $\nexists \lim\limits_{x\to 0} f_1$, entonces $\nexists \lim\limits_{x\to 0} f(x)$

- técnicas resolución límite
  - → Probar que no existen → probar por curvas
  - → Probar 0 → probar acotado
  - → Cambios de variable
  - → simplificar
  - → parametrización

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(g) $$\lim_{(x,y)\to(0,0)} \left(\underbrace{y\cos(x^{-1})}_{f_1}, \underbrace{\frac{\operatorname{sen}(3x)}{2x}}_{f_2}, \underbrace{x-y}_{f_3} \right) = \lim_{(x,y)\to(0,0)} \left(f_1(x,y), f_2(x,y), f_3(x,y)\right)$$

• $\lim_{(x,y)\to(0,0)} \overset{\to 0}{y}\, \overset{\to\infty}{\cos(x^{-1})} = 0$

acotado en [0,1]

• $\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(3x)}{2x} = \lim_{(x,y)\to(0,0)} \frac{\cos(3x)}{2} = \frac{1}{2}$

L'Hop

• $\lim_{(x,y)\to(0,0)} x-y = 0$

$$\lim_{(x,y)\to(0,0)} \left(y\cos(x^{-1}), \frac{\operatorname{sen}(3x)}{2x}, x-y\right) = \left(0, \frac{1}{2}, 0\right)$$

(h) $$\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(x^2-y^2)}{x-y} = \text{Ind}$$

• Probar con recta $y=mx$

$$\lim_{x\to0} \frac{\operatorname{sen}(x^2-m^2x^2)}{x-mx} \overset{L'H}{=} \lim_{x\to0} \frac{\cos(x^2-m^2x^2)(2x-2xm^2)}{1-m} = 0$$

• Probar parábolas $y=kx^2$

$$\lim_{x\to0} \frac{\operatorname{sen}(x^2-k^2x^4)}{x-kx^2} = \lim_{x\to0} \frac{\cos(x^2-k^2x^4)}{1-k} \cdot (2x-4x^3k^2) = 0$$

$$\frac{\operatorname{sen}(x^2-y^2)}{x-y} = \frac{\operatorname{sen}((x+y)(x-y))}{(x-y)} = \frac{\operatorname{sen}(x^2-y)(x+y)}{(x-y)(x+y)}$$

$$\left(\frac{\operatorname{sen}(x^2-y)}{x^2-y^2}\right)x + \left(\frac{\operatorname{sen}(x^2-y)}{x^2-y^2}\right)\cdot y$$

$$\lim_{x,y\to0,0} \left[\frac{\operatorname{sen}(x^2-y)}{x^2-y}\, x + \frac{\operatorname{sen}(x^2-y^2)}{x^2-y^2}\, y\right] = 0$$

A chequear

(I) $$\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2} \quad \text{Ind } \frac{0}{0}$$

$$\boxed{u = x^2+y^2 \quad \lim_{x,y\to0,0} u = 0}$$

$$\lim_{u\to0} \frac{e^u-1}{u} = \lim_{u\to0} \frac{e^u}{1} = 1$$

(J) $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \overset{\to0}{x} \cdot \frac{y^2}{x^2+y^2}$$

acotado en $B^*((0,0),\varepsilon)$

(k) $$\lim_{(x,y)\to(0,0)} \frac{y^5}{x^2+y^4} = \text{ind. } \frac{0}{0} \Rightarrow \lim_{(x,y)\to(0,0)} \overset{\to0}{y}\, \frac{y^4}{x^2+y^4} = 0$$

acotado $B^*((0,0),\varepsilon)$

$y^4 \le x^2+y^2$

$\left|\frac{y^4}{x^2+y^2}\right| \le 1 \to \left|\frac{y^5}{x^2+y^2}\right| \le |y|$

• analizar curva $y=mx$

$$\lim_{x\to0} \frac{(mx)^3}{x^2+(mx)^4} = \lim_{x\to0} \frac{x^3m^3}{x^2(1+m^4x^2)} = \lim_{x\to0} \frac{x^3m^3}{(1+m^4x^2)} \overset{L'H}{=} \lim_{x\to0} \frac{3\cancel{x^2}m^3}{2m^4\cancel{x}} = \lim_{x\to0} \frac{3xm^3}{2m^4} = 0$$

• analizar $x=ky^2$

$$\lim_{y\to0} \frac{y^5}{k^2y^4+y^4} = \lim_{y\to0} \frac{y^5}{y^4(k^2+1)} = \lim_{y\to0} \frac{y}{k^2+1} = 0$$

(l) $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(\overbrace{x^2+y^2}^{\geq 0\ \forall x \wedge y}) = \lim_{u\to 0} \overset{\nearrow 0}{u}\underbrace{\ln(u)}_{\to\infty} = \text{Ind } 0\cdot\infty$

$$\boxed{\begin{array}{l} u = x^2+y^2 \\ \lim_{x,y\to 0,0} u = 0 \end{array}}$$

$\lim_{u\to 0} u\ln(u) = \lim_{u\to 0} \ln(u^u)$

$\lim_{u\to 0} \ln\left[\underbrace{\left(1+\frac{u^2-u}{u}\right)^{u\cdot\frac{1}{u^2-u}}}_{e}\right] = 1$

$\lim_{u\to 0} u\ln u = \lim_{u\to 0} \frac{\ln(u)^{\to\infty}}{\left(\frac{1}{u}\right)_{\to\infty}} \overset{L'H}{=} \lim_{u\to 0} \frac{\left(\frac{1}{u}\right)}{-\left(\frac{1}{u^2}\right)}$

$= \lim_{u\to 0} -\frac{1}{u}\ \frac{u^2}{1} = 0$

(m) $\lim_{(x,y)\to(1,-1)} \frac{\overset{\to 0}{x^3+y^3}}{x+y_{\to 0}}$ $\Rightarrow$ división polinómica

$$\begin{array}{r|l} x^3+y^3 & x+y \\ -\ x^3+yx^2 & x^2+y^2-yx \\ \hline y^3-yx^2 & \\ -\ y^3+y^2x & \\ \hline -yx^2-y^2x & \\ -\ -yx-y^2x & \\ \hline 0/ & \end{array}$$

$= \lim_{(x,y)\to(1,-1)} \underbrace{x^2+y^2}_{2}-\underbrace{yx}_{1} = 3$

(n) $\lim_{(x,y)\to(1,1)} \frac{\overset{\to 0}{2y^2-x^2-xy}}{y-x_{\to 0}} = \lim_{(x,y)\to(1,1)} \frac{y^2+y^2-x^2-xy}{(y-x)} = \lim_{x,y\to(1,1)} \frac{(y-x)(y+x)+y(y-x)}{(y-x)} = \lim_{x,y\to(1,1)} y+x+y = 3$

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* Ejercicio 6 : $f = D\subset\mathbb{R}^n \to \mathbb{R}$, $n>1$ A punto de acum de D, $\lim_{X\to A} f(x) = L$

Curva C contiene A y $C-\{A\}\subset D$, $\lim_{X\overset{C}{\to}A} f(x) = L_c$, Demo $L_c = L$
con $X\in C-\{A\}$

- Analizar condiciones

$\Rightarrow \lim_{X\to A} f(x) = L$ existe límite de función de $f(x)$ con x tendiendo a A

$\Rightarrow \lim_{X\overset{C}{\to}A} f(x) = L_c \to$ existe límite en $f(x)$ en A si me acerco con curva C

$\Rightarrow$ los puntos de la curva menos A es del dominio $C-\{A\}\subset D$

- Resolución:

Me dicen que $C-\{A\}\subset D$, entonces si hago $\lim_{X\to A} f(x)$, con $X\in C-\{A\}$, entonces $X\in \underbrace{C-\{A\}}_{\in D}$
los puntos $X\in C-\{A\}\subset D$ que utilizo para evaluar lím pertenecen al Dom de D y al dom de la curva C por lo que $\lim_{X\to A} f(x) = \lim_{X\overset{C}{\to}A} f(x)$ $\forall\ X\in C-\{A\}$

* **Ejercicio 7** = Analizar límites

(a) $\lim_{(x,y)\to(0,0)} \dfrac{y\,\text{sen}(x)}{x^2+y^2}$ $\qquad \nexists \lim_{(x,y)\to(0,0)} \dfrac{y\,\text{sen}(x)}{x^2+y^2}$

Como da distinto valor de lim con distintas rectas que pasan ahí no existe lim

- Probar con rectas que pasan $(0,0)$ $y=mx$

$$\lim_{x\to 0} \frac{mx\,\text{sen}(x)}{x^2+(mx)^2} = \lim_{x\to 0} \frac{x\; m\,\text{sen}(x)}{x^2(1+m^2)} = \lim_{x\to 0}\frac{m\,\text{sen}(x)}{x(1+m^2)} \overset{L'H}{=} \lim_{x\to 0} \frac{m\cos(x)}{1+m^2} = \begin{cases} \frac{m}{1+m^2} & \text{si } m>0 \\ -\frac{m}{1+m^2} & \text{si } m<0 \\ 0 & \text{si } m=0 \end{cases}$$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} x\cdot\dfrac{y^2}{x^2+y^4}$ $\qquad$ no sirve acotar $\underbrace{y^2\le y^4}_{\text{valido si } y>1}$ $\qquad \nexists \lim_{x,y\to 0,0} \dfrac{xy^2}{x^2+y^4}$

- Probar con parábolas $x = ay^2$

$$\lim_{y\to 0} \frac{ay^2\,y^2}{(ay^2)^2+y^4} = \lim_{y\to 0}\frac{ay^4}{a^2y^4+y^4} = \lim_{y\to 0}\frac{ay^4}{y^4(a^2+1)} = \lim_{y\to 0}\frac{a}{a^2+1} = \begin{cases} \frac{a}{a^2+1} & \text{si } a>1 \\ -\frac{a}{a^2+1} & \text{si } a<1 \\ 0 & \text{si } a=0\end{cases}$$

con $a=2 \to \lim = \frac{2}{5}$

$a=-2 \to \lim = \frac{-2}{5}$

$a=0 \to \lim = 0$

si lim existe es único

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* **Ejercicio 8** $\quad f(x,y) = \dfrac{x^2+2x}{2x^2+y^2} \Rightarrow$ D dom nat y $N_k(f)$ conj nivel. Hallar N_0, $N_{\frac{1}{4}}(f)$, $N_1(f)$

- Hallar dominio natural

$\underbrace{2x^2+y^2}_{\geq 0\ \forall x \wedge y} \neq 0$

$2x^2+y^2=0$ si $x=0 \wedge y=0$

$D = \{(x,y)\in\mathbb{R}^2\} - \{(0,0)\}$

$(0,0)$ no pertenece Dom es pto acumulación

- Hallar $N_0(f) = \{(x,y)\in D \,/\, x=0\} \cup \{(x,y)\in D \,/\, x=-2\}$

$f(x,y) = \dfrac{x^2+2x}{2x^2+y^2} = 0 \qquad x^2+2x=0$

$x(x+2)=0 \qquad \underbrace{x=0 \vee x=-2}_{\text{rectas verticales en } \mathbb{R}^2}$

- Hallar $N_1 = \{(x,y)\in D \,/\, (x-1)^2-y^2=1\}$

$f(x,y) = \dfrac{x^2+2x}{2x^2+y^2} = 1$

$x^2+2x = 2x^2+y^2$

$-x^2+2x = y^2$

$x^2-2x-y^2=0$

$(x-1)^2-1-y^2=0$

$(x-1)^2-y^2=1$ hipérbole $c=(1,0)$

- Hallar $N_{\frac{1}{4}}(f) = \{(x,y)\in D \,/\, \frac{(x+2)^2}{2} - \frac{y^2}{8} = 1\}$

$f(x,y) = \dfrac{x^2+2x}{2x^2+y^2} = \dfrac{1}{4}$

$x^2+2x = \frac{1}{2}x^2+\frac{1}{4}y^2 \Rightarrow \frac{1}{2}x^2+2x = \frac{1}{4}y^2$

$\frac{1}{2}(x^2+2.2.x) = \frac{1}{4}y^2 \Rightarrow \frac{1}{2}(x^2+2.2x+2^2-2^2) = \frac{1}{4}y^2$

$\frac{1}{2}[(x+2)^2-4] = \frac{1}{4}y^2$

$(x+2)^2 - \frac{y^2}{2} = 2$

$\frac{(x+2)^2}{2} - \frac{y^2}{4} = 1$ Hipérbola $c=(-2,0)$

$\Rightarrow$ reemplazando cada curva de conj nivel, vemos que todas las curvas pasan por $(0,0)$,

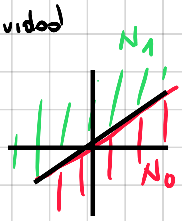
$(0,0)$ está en el todos los conjuntos $\to (0,0)\in N_1, \in N_0, N_{\frac{1}{4}}$

por lo que $\lim_{x,y\to(0,0)} f(x)$ tendrá un límite distinto dependiendo cuál curva se tome

f no se aproxima a un solo valor

* Ejercicio 9 : Determinar puntos de continuidad

ⓐ $f(x,y) = \begin{cases} 1 & \text{cuando } x-y>0 \\ 0 & \text{cuando } x-y\leq 0 \end{cases}$



puntos continuos $\{(x,y)\in\mathbb{R}^2 / x>y\} \cup \{(x,y)\in\mathbb{R}^2 / x<y\}$

puntos discontinuos $\{(x,y)\in\mathbb{R}^2 / x=y\}$

- $f(x,x) = 0 \Rightarrow \exists f(x,x)$
- $\lim\limits_{x\to y} f(x,y) = \left\{\begin{array}{l} \lim\limits_{x\to y^+} f(x) = 1 \\ \lim\limits_{x\to y^-} f(x) = 0 \end{array}\right\} \nexists \lim\limits_{x\to y} f(x,y)$

ⓑ $f(x,y) = \left\{\begin{array}{lll} x^2-y & \text{si} & x\neq 2y \\ 3 & \text{si} & x=2y \end{array}\right\}$ probar continuidad en $(2b,b)$ punto genérico $\Big] \to$ continua si $\lim\limits_{x,y\to a,b} f(x,y) = f(a,b)$

- $f(2b,b) = 3 \to$ existe imagen $\forall\, b\in\mathbb{R}$
- $\lim\limits_{(x,y)\to(2b,b)} x^2-y = (2b)^2-b = \underline{4b^2-b} \nearrow$ si $b=1 \to \lim\limits_{x,y\to 2b,b} f(v) = 15 \to$ varía según valores de b

puntos de discontinuidad $\{(x,y)\in\mathbb{R}^2 / x=2y\}$

puntos de continuidad $\{(x,y)\in\mathbb{R}^2 / x\neq 2y\}$

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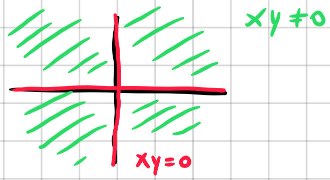
ⓒ $f(x,y) = \begin{cases} 0 & \text{cuando } xy\neq 0 \to x\neq 0 \wedge y\neq 0 \\ 1 & \text{cuando } xy=0 \to x=0 \vee y=0 \end{cases}$



puntos discontinuos
$\{(x,y)\in\mathbb{R}^2 / x=0 \vee y=0\}$

puntos continuos
$\{(x,y)\in\mathbb{R}^2 / x\neq 0 \wedge y\neq 0\}$

- Continuidad en $(a,0)$
  - $f(a,0) = 1$
  - $\lim\limits_{(x,y)\to(a,0)} f(x,y) = 0$

  $\Big\}$ discontinuidad

- Continuidad $(0,b)$
  - $f(0,b) = 1$
  - $\lim\limits_{(x,y)\to(0,b)} f(x,y) = 0$

  $\Big\}$ disc

ⓓ $f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{si } (x,y)\neq(0,0) \\ 0 & \text{si } (x,y)=(0,0) \end{cases}$

→ Estudiar cont. $(x,y)=(0,0)$

- $f(0,0) = 0 \to$ existe img

$\underline{f(0,0) = \lim\limits_{(x,y)\to(0,0)} f(x,y)}$
f continua en $(0,0)$

- $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \overset{\nearrow 0}{y} \cdot \overset{\nearrow \text{acotada}}{\dfrac{x^2}{x^2+y^2}} = 0$

puntos continuos $\{(x,y)\in\mathbb{R}^2\}$

ptos discontinuos $\emptyset$

probar acotada $B^*((0,0),\varepsilon)$

$\underbrace{x^2}_{\geq 0} \leq \underbrace{x^2+y^2}_{\geq 0} \to \underbrace{\dfrac{x^2}{x^2+y^2}}_{\geq 0} \leq 1 \Rightarrow \left|\dfrac{x^2}{x^2+y^2}\right| \leq \dfrac{1}{M}$

ⓔ $f(x,y) = \begin{cases} \dfrac{(x-2)^2}{(x-2)^2+y^2} & \text{si } (x,y) \neq (2,0) \\ 0 & \text{si } (x,y) = (2,0) \end{cases}$ $\Rightarrow$ Ver continuidad en $(x,y)=(2,0)$

puntos continuos $\{(x,y)\in\mathbb{R}^2\} - \{(2,0)\}$

puntos discontinuos $\{(2,0)\}$

$\rightarrow f(2,0) = 0 \rightarrow$ existe imagen

$\rightarrow \lim\limits_{(x,y)\to(2,0)} \dfrac{(x-2)^2}{(x-2)^2+y^2}$ Ind $\frac{0}{0}$

$(x-2)^2 \leq (x-2)^2+y^2$

$\dfrac{(x-2^2)}{(x-2)^2+y^2} \leq 1$ acotado $B^*((2,0),\varepsilon)$

probar rectas $y = m(x-2)$

$\lim\limits_{x\to 2} \dfrac{(x-2)^2}{(x-2)^2 - m^2(x-2)^2} = \dfrac{1}{1-m^2}$

si $m=2$ entonces $\lim = \frac{-1}{3}$

si $m=3$ entonces $\lim = \frac{-1}{8}$

límites distintos, $\nexists \lim\limits_{(x,y)\to(2,0)} f(x,y)$

ⓕ $f(x,y) = \begin{cases} x & \text{si } 4x^2+y^2 < 1 \\ 2x+y & \text{si } 4x^2+y^2 \geq 1 \end{cases}$ $\Rightarrow$ verificar continuidad en puntos $\boxed{4x^2+y^2=1}$

- Hallar puntos donde $4x^2+y^2=1$

$y^2 = 1-4x^2 \rightarrow \underbrace{y=\sqrt{1-4x^2}}_{\text{punto } (a,\sqrt{1-4a^2})} \vee \underbrace{y=-\sqrt{1-4x^2}}_{(a,-\sqrt{1-4a^2})}$, $\overbrace{1-4x^2 \geq 0 \rightarrow \frac{1}{4} \geq x^2}^{\text{Hallar dominio } x}$ $\boxed{\frac{-1}{2} \leq x \leq \frac{1}{2}}$

- Probar continuidad para puntos $(a,\sqrt{1-4a^2})$

$\rightarrow f(a,\sqrt{1-4a}) = 2a+\sqrt{1-4a} \Rightarrow$ existe imagen

$\rightarrow \lim\limits_{(x,y)\to(a,\sqrt{1-4a})} f(x,y) = \begin{cases} \lim\limits_{(x,y)\to(a,\sqrt{1-4a})^+} f(x,y) = 2a+\sqrt{1-4x} \\ \lim\limits_{(x,y)\to(a,\sqrt{1-4a})^-} f(x,y) = a \end{cases}$

$\lim\limits_{(x,y)\to(a,\sqrt{1-4a})^+} f(x,y) \neq \lim\limits_{(x,y)\to(a,\sqrt{1-4a})^-} f(x,y)$

$\downarrow$

no existe lim $\rightarrow$ no continuo en $(a,\sqrt{1-4a})$

- Probar continuidad en $(a,-\sqrt{1-4a})$

$\rightarrow f(a,-\sqrt{1-4a}) = 2a-\sqrt{1-4a}$

$\rightarrow \lim\limits_{(x,y)\to(a,-\sqrt{1-4a})} f(x) = \begin{cases} \lim\limits_{(x,y)\to(a,-\sqrt{1-4a})^+} f(x,y) = 2a-\sqrt{1-4a} \\ \lim\limits_{(x,y)\to(a,\sqrt{1-4a})^-} f(x,y) = a \end{cases}$

WTF

* Ejercicio 10 = Hallar, si es posible, k / f sea continua en $\mathbb{R}^2$

$$f(x,y) = \begin{cases} \dfrac{x^3 - x(y+1)^2}{x^2+(y+1)^2} & \text{si } (x,y) \neq (0,-1) \\ k & \text{si } (x,y) = (0,-1) \end{cases}$$

→ $f(0,-1) = k$ → existe imagen → Hallar $k \in \mathbb{R}$ / $\lim_{(x,y)\to(0,-1)} f(x,y) = f(0,-1) = k$

→ $\lim_{(x,y)\to(0,-1)} \dfrac{\overset{\to 0}{x^3 - x(y+1)^2}}{\underset{\to 0}{x^2+(y+1)^2}} = \text{Ind } \frac{0}{0}$

- Intentar transformar expresión

$$\frac{x^3-x(y+1)^2}{x^2+(y+1)^2} = \frac{x[x^2-(y+1)^2]}{x^2+(y+1)^2} = x\underbrace{\left(\frac{x^2-(y+1)^2}{x^2+(y+1)^2}\right)}$$

División polinómica

$$\begin{array}{l|l} x^2-(y+1)^2 & x^2+(y+1)^2 \\ \underline{-x+(y+1)^2} & 1 \\ 0-2(y+1) & \end{array}$$

$$x\left(-\frac{2(y+1)^2}{x^2+(y+1)^2}+1\right)$$

$$x\left[\frac{(x-(y+1))(x+(y+1))}{x^2+(y+1)^2}\right] \Rightarrow \text{No sirve simplificar expr}$$

$$\lim_{(x,y)\to(0,-1)} \frac{x^3-x(y+1)^2}{x^2+(y+1)^2} = \lim_{(x,y)\to(0,-1)} x\left(\frac{x^2-(y+1)^2}{x^2+(y+1)^2}\right) = \lim_{(x,y)\to(0,-1)} x\left(\frac{x^2}{x^2+(y+1)^2} - \frac{(y+1)^2}{x^2+(y+1)^2}\right)$$

$$= \lim_{(x,y)\to(0,-1)} \overset{\to 0}{x}\cdot\underbrace{\frac{x^2}{x^2+(y+1)^2}}_{\substack{\text{acotado} \\ B^*((0,-1),\varepsilon)}} - \lim_{(x,y)\to(0,-1)} \overset{\to 0}{x}\underbrace{\frac{(y+1)^2}{x^2+(y+1)^2}}_{\text{acotado}} = 0-0 = 0 \to \boxed{k=0}$$

DERIVADA FUNCIONES VARIAS VARIABLES

* Ejercicio 10 = Hallar derivadas parciales y evaluar en punto

(a) $f(x,y) = xy + x^2$, en $(2,0)$

~~rect tg x ⇒ $f(2,0) + \overset{4}{\widetilde{\ \ }}(x-2)$~~

- $\frac{\partial f}{\partial x}(x,y) = y + 2x$, $\frac{\partial f}{\partial x}(2,0) = 4$
- $\frac{\partial f}{\partial y}(x,y) = x$, $\frac{\partial f}{\partial y}(2,0) = 2$

(b) $f(x,y) = \operatorname{senh}(x^2+y)$, en $(1,-1)$

- $\frac{\partial f}{\partial x}(x,y) = \cosh(x^2+y)\cdot 2x$, $\frac{\partial f}{\partial x}(1,-1) = \cosh(0)\cdot 2 = 2$
- $\frac{\partial f}{\partial y}(x,y) = \cosh(x^2+y)$, $\frac{\partial f}{\partial y}(1,-1) = \cosh(0) = 1$