23/6/2025

Teoremas Integrales

*** Ejercicio 2:** $f(x,y,z) = (2x - y^2 z,\ h(x,y))$, hallar $h(x,y)$ para f irrotacional y $\vec{f}(\vec{0}) = (0,0,1)$

- Condición $\vec{f}(\vec{0}) = (0,0,h(0,0)=1) \longrightarrow h(0,0) = 1$
- Condición $\vec{f}$ irrotacional $\rightarrow$ $rot(\vec{f}) = \nabla \times f = 0$

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -y^2 z & h(x,y) \end{vmatrix} = \left(\frac{\partial h}{\partial y} + y^2,\ -\frac{\partial h}{\partial x},\ 0 \right) = (0,0,0)$$

$$\begin{cases} \frac{\partial h}{\partial y} + y^2 = 0 \\ -\frac{\partial h}{\partial x} = 0 \end{cases} \rightarrow \quad h'_y = -y^2 \qquad \underbrace{h = -\frac{y^3}{3} + k}_{\frac{\partial h}{\partial x} = 0 \ \checkmark} \qquad \boxed{h = \frac{-y^3}{3} + 1}$$

$$dh = -y^2\, dy$$

$$\int 1\, dh = -\int y^2 dy \qquad h(0,0) = -\frac{y^3}{3} + k = 1 \rightarrow \boxed{k = 1}$$

*** Ejercicio 3** $\vec{f}(x,y,z) = (y^2 - z,\ 2xy + 1,\ h(x,y,z))$, ¿posible hallar h / $\vec{f}$ selenoidal e irrotacional?

- irrotacional $= rot(f) = 0$

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z & 2xy+1 & h \end{vmatrix} = \left(\frac{\partial h}{\partial y},\ -1 - \frac{\partial h}{\partial x},\ 0 \right) = (0,0,0) \qquad \begin{cases} \frac{\partial h}{\partial y} = 0 \\ -1 - \frac{\partial h}{\partial x} = 0 \end{cases}$$

- hallar h $\qquad\qquad \rightarrow$ hallar α_1

$$\frac{dh}{dx} = 1 \qquad h = x + \alpha_1(y) + \alpha_2(z) + k \qquad \frac{dh}{dy} = \alpha_1'(y) = 0 \rightarrow \alpha_1(y) = k_1$$

$$dh = dx$$

- selenoidal $= div(f) = 0$

$$div(\vec{f}) = \nabla \cdot f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (y^2 - z,\ 2xy+1,\ h) = 0 + 2x + \frac{\partial h}{\partial z} = 0 \rightarrow \frac{\partial h}{\partial z} = -2x$$

$\rightarrow$ hallar α_2

$$\frac{\partial h}{\partial z} = \alpha_2'(z) = -2x$$

$$\int d\alpha_2 = \int -2x\, dz \rightarrow \alpha_2 = -2xz$$

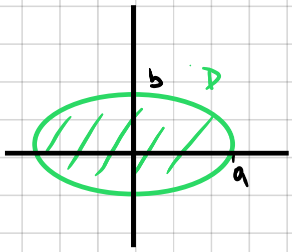
$$h = x - 2xz \longrightarrow \text{no cumple con condiciones,}$$

no es posible hallar h

* **Ejercicio 4** Area $D = \frac{1}{k} \oint_{\partial D} \vec{f} \cdot d\vec{s}$, (Teorema de Green)

↳ Graficar D, y calcular área cuando $\vec{f}(x,y) = (0,x)$ sobre ∂D

(a) $D = \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ → no es posible usar coord polares para Integrales dobles para hallar área(D) ya que r varía según semieje



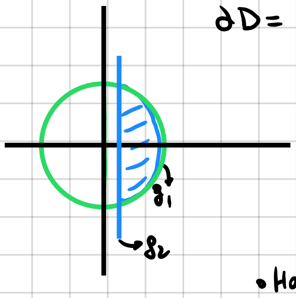
$\partial D = \vec{g}(t) = (a\cos(t), b\,\text{sen}(t))$ $0 \leq t \leq 2\pi$ $\vec{f} = (0,x)$ $g'(t) = (-a\,\text{sen}(t), b\cos(t))$

$$\text{Area}(D) = \oint_{\partial D^+} \vec{f}(\vec{g}(t)) \cdot \vec{g}'(t)\, dt$$

$$= \int_0^{2\pi} (0, a\cos(t)) \cdot (-a\,\text{sen}(t), b\cos(t))\, dt = \int_0^{2\pi} ab\cos^2(t)\, dt = ab\int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos(2t)}{2}\right) dt$$

$$= \frac{ab}{2}\left(t + \frac{\text{sen}(2t)}{2} \Big|_0^{2\pi} \right) = \frac{ab}{2}\left[(2\pi + 0) - (0)\right] = ab\pi$$

(b) $D = x^2 + y^2 \leq 2$, $x \geq 1$



$\partial D = D_1 + D_2$ →

$\vec{g}_1(t) = (\sqrt{2}\cos(t), \sqrt{2}\,\text{sen}(t))$, $-\frac{\pi}{4} \leq t \leq \frac{\pi}{4}$

$\sqrt{2}\cos(t) \geq 1$
$t \geq \arccos\left(\frac{1}{\sqrt{2}}\right)$

$\vec{g}_2(t) = (1, t)$ $1 \geq t \geq -1$

• Hallar derivadas

$$g_1'(t) = (-\sqrt{2}\,\text{sen}(t), \sqrt{2}\cos(t)) , \quad g_2'(t) = (0,1)$$

• Hallar $\oint_{\partial D} \vec{f}\, d\vec{s}$

$$↳ D_1 = \int_{D_1^+} \vec{f}\, d\vec{s} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (0, \sqrt{2}\cos(t))(-\sqrt{2}\,\text{sen}(t), \sqrt{2}\cos(t))\, dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 + \cos(2t)\, dt = \left(t + \frac{\text{sen}(2t)}{2}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\right)$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\right) - \left(-\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} + 1$$

$$↳ D_2 : \int_{D_2^+} \vec{f}\, d\vec{s} = \int_{-1}^{1} (0,1)(0,1)\, dt = \int_{-1}^{1} 1\, dt = t\Big|_{-1}^{1} = 1 - (-1) = 2$$

$$D_1 + D_2 = \left(\frac{\pi}{2} + 1\right) + (2) = \frac{\pi}{2} + 3$$

* Ejercicio 5 $\quad D = \{(x,y) \in \mathbb{R}^2 / 4x^2+y^2 \leq 16\}$ $\quad f(x,y) = (g(x)y, g(x)+y^2)$, $\vec{f} \in C^1 / f(0,1) = (1,2)$

↳ Hallar expresión de g para $\oint_{\partial D^+} \vec{f} \cdot d\vec{s} = \iint_D (Q'_x - P'_y)\, dxdy = \text{Area}(D)$

∘ Evaluar condiciones

↳ $Q'_x - P'_y = 1$ para que $\oint_{\partial D^+} \vec{f}\, d\vec{s} = \iint_D 1\, dxdy = \text{Area}(D)$

↳ $\vec{f}(0,1) = (g(0), g(0)+1) = (1,2) \longrightarrow g(0) = 1$

• Hallar $Q'_x - P'_y = 1$

$P = g(x)y$, $Q = g(x)+y^2$

$P'_y = g(x)$, $Q'_x = g'(x)$

$g'(x) - g(x) = 1$

$\frac{dg}{dx} = 1+g$

$\int \frac{dg}{1+g} = \int dx$

$\ln|1+g| = x + k$

$|1+g| = e^x \cdot \underbrace{e^k}_{>0}$

$g = ce^x - 1$, $c \in \mathbb{R}$

∘ Hallar $c \longrightarrow g(0) = c \cdot e^0 - 1 = 1 \longrightarrow c = 2$ $\qquad g = 2e^x - 1$

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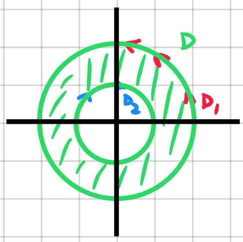
* Ejercicio 6 $\quad f(x,y) = (x, xy^2)$, $\quad D: 1 \leq x^2+y^2 \leq 4$

(a) Calc circulación $\vec{f}$ con integral de línea en ∂D



$\oint_{\partial D^+} \vec{f}\, d\vec{s} = \oint_{D_1^+} \vec{f}\, d\vec{s} + \oint_{D_2^+} \vec{f}\, d\vec{s}$

$D_1 =$ $\quad \vec{g}_1(t) = (2\cos(t), 2\text{sen}(t))$ $\quad 0 \leq t \leq 2\pi$

$\vec{g}_1'(t) = (-2\text{sen}(t), 2\cos(t))$

$\vec{f}(g_1(t)) = (2\cos(t), 2\cos(t)4\text{sen}^2(t))$

$D_2:$ $\quad \vec{g}_2(t) = (\text{sen}(t), \cos(t))$ $\quad 0 \leq t \leq 2\pi$

$\vec{g}_2'(t) = (\cos(t), -\text{sen}(t))$

$\vec{f}(\vec{g}_2(t)) = (\text{sen}(t), \text{sen}(t)\cos^2(t))$

$$\oint_{D_1^+} \vec{f}\, d\vec{s} = \int_0^{2\pi} (2\cos(t), 2\cos(t)4\text{sen}^2(t))(-2\text{sen}(t), 2\cos(t))\, dt = \int_0^{2\pi} -2\text{sen}(2t) + \text{sen}^2(2t)\, dt$$

$$= \int_0^{2\pi} -2\text{sen}\, 2t + 1 - \left[\frac{1}{2} + \cos(2t)\right] dt = \cos(2t) + \frac{1}{2}t + \frac{\text{sen}(2t)}{2}\Big|_0^{2\pi} = (1+\pi+0) - (1+0+0) =$$

wolfram 4π

$$\oint_{D_2^+} \vec{f}\, d\vec{s} = \int_0^{2\pi} (\text{sen}(t), \text{sen}(t)\cos^2(t))(\cos(t), -\text{sen}(t))\, dt = \int_0^{2\pi} \frac{1}{2}\text{sen}(2t) - \frac{1}{4}\text{sen}^2(2t)\, dt =$$

$\longrightarrow$ wolfram $= -\frac{\pi}{4}$

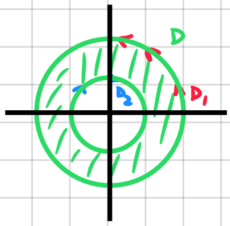
$$\int_0^{2\pi} \frac{1}{2}\text{sen}(2t) - \frac{1}{4}\left(\frac{1}{2} + \frac{\cos(4t)}{2}\right) dt = -\frac{1}{4}\cos(2t) - \frac{1}{}$$

$$\frac{1}{4}\left(\frac{1}{2} - \frac{\cos(4t)}{2}\right) dt = -\frac{1}{4}$$

$$\oint_{\partial D^+} \vec{f}\, d\vec{s} = 4\pi - \frac{\pi}{4} = \frac{15\pi}{4}$$

* Ejercicio 6 $\quad f(x,y)=(x,xy^2)$, $\quad D: 1\le x^2+y^2\le 4$

ⓐ Calc circulación $\vec{f}$ con integral de línea en ∂D



$$\oint_{\partial D^+} \vec{f}\,d\vec{s} = \oint_{D_1^+} \vec{f}\,d\vec{s} + \oint_{D_2^+} \vec{f}\,d\vec{s}$$

$D_1 =$ $\quad \vec{g}_1(t)=(2\cos(t),2\operatorname{sen}(t))$ $\quad 0\le t\le 2\pi$

$\vec{g}_1'(t)=(-2\operatorname{sen}(t),2\cos(t))$ $\quad \vec{f}(\vec{g}_1(t))=(2\cos(t),\ 2\cos(t)4\operatorname{sen}^2(t))$

$D_2:$ $\quad \vec{g}_2(t)=(\operatorname{sen}(t).\cos(t))$ $\quad 0\le t\le 2\pi$

$\vec{g}_2'(t)=(\cos(t),-\operatorname{sen}(t))$ $\quad \vec{f}(\vec{g}_2(t))=(\operatorname{sen}(t),\ \operatorname{sen}(t)\cos^2(t))$

$$\oint_{D_1^+}\vec{f}\,d\vec{s}=\int_0^{2\pi}(2\cos(t),2\cos(t)4\operatorname{sen}^2(t))(-2\operatorname{sen}(t),2\cos(t))\,dt=\int_0^{2\pi}-2\operatorname{sen}(2t)+\operatorname{sen}^2(2t)\,dt$$

$$=\int_0^{2\pi}-2\operatorname{sen}2t+4\left[\frac{1}{2}+\cos(2t)\right]dt=\cos(2t)+\frac{1}{2}t+\frac{\operatorname{sen}(2t)}{2}\Big|_0^{2\pi}=(1+\pi+0)-(1+0+0)=$$

wolfram 4π

$$\oint_{D_2^+}\vec{f}\,d\vec{s}=\int_0^{2\pi}(\operatorname{sen}(t),\operatorname{sen}(t)\cos^2(t))(\cos(t),-\operatorname{sen}(t))\,dt=\int_0^{2\pi}\frac{1}{2}\operatorname{sen}(2t)-\frac{1}{4}\operatorname{sen}^2(2t)\,dt=$$

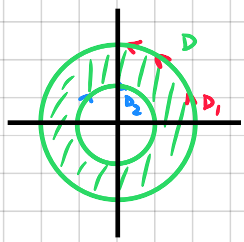
$$\int_0^{2\pi}\frac{1}{2}\operatorname{sen}(2t)-\frac{1}{4}\left(\frac{1}{2}+\frac{\cos(4t)}{2}\right)dt=-\frac{1}{4}\cos(2t)-\frac{1}{}$$

$$\frac{1}{4}\left(\frac{1}{2}-\frac{\cos(4t)}{2}\right)dt=-\frac{1}{4}$$

wolfram $=-\frac{\pi}{4}$

$$\oint_{\partial D^+}\vec{f}\,d\vec{s}=4\pi-\frac{\pi}{4}=\frac{15\pi}{4}$$

ⓐ Calc circulación $\vec{f}$ con teorema de Green



$$\oint_{\partial D^+}\vec{f}\,d\vec{s}=\iint_D (Q_x'-P_y')\,dx\,dy$$

D en coord polares

$\left\{\begin{array}{l} x=r\cos(\theta)\\ y=r\operatorname{sen}(\theta)\end{array}\right.$, $|J|=r$ $\quad 1\le r\le 2$ $\quad 0\le\theta\le 2\pi$

- Hallar $Q_x'-P_y'$. $\{P=x,\ Q=xy^2\}$

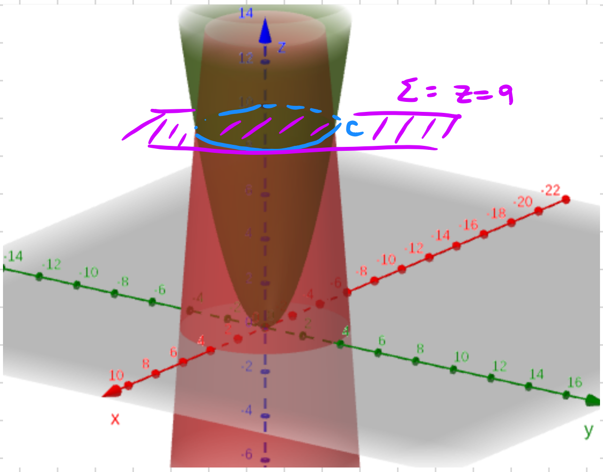
$Q_x'=y^2 \quad P_y'=0 \qquad Q_x'-P_y'=y^2$

$$\oint_{\partial D^+}\vec{f}\,d\vec{s}=\iint_D y^2\,dx\,dy=\int_1^2\int_0^{2\pi} r^2\operatorname{sen}^2(\theta)\,r\,d\theta\,dr=\int_1^2 r^3\int_0^{2\pi}\frac{1}{2}-\cos(2\theta)\,d\theta\,dr$$

$$=\int_1^2 r^3\left(\frac{1}{2}\theta-\frac{\operatorname{sen}(2\theta)}{2}\Big|_0^{2\pi}\right)dr=\int_1 r^3(\pi)\,dr=\frac{\pi r^4}{4}\Big|_1^2=4\pi-\frac{\pi}{4}=\frac{15\pi}{4}$$

* Ejercicio 8 $\vec{f}(x,y,z) = (y+x,\ 2y,\ z)$ $C: \begin{cases} z = 27 - 2x^2 - 2y^2 \\ z = x^2 + y^2 \end{cases}$

ⓐ graficar y parametrizar C



parametrizar C

$C: \begin{cases} z = 27-2x^2-2y^2 \\ z = x^2+y^2 \end{cases} \rightarrow 27-2x^2-2y^2 = x^2+y^2$

$3x^2+3y^2 = 27 \qquad \underline{x^2+y^2 = 9 = 3^2}$

hallar valor z de intersección $\boxed{z=9}$ → un plano

$\vec{g}(t) = (3\cos(\theta),\ 3\operatorname{sen}(\theta),\ 3)\ ,\quad 0 \leq \theta \leq 2\pi$

$\vec{f}(\vec{g}(t)) = (3\operatorname{sen}(\overset{t}{\theta}) + 3\cos(\overset{t}{\theta}),\ 6\operatorname{sen}(\overset{t}{\theta}),\ 3)$

$\vec{g}'(t) = (-3\operatorname{sen}(\overset{t}{\theta}),\ 3\cos(\overset{t}{\theta}),\ 0)$

ⓑ Calcular circulación $\vec{f}$ por C con Int línea

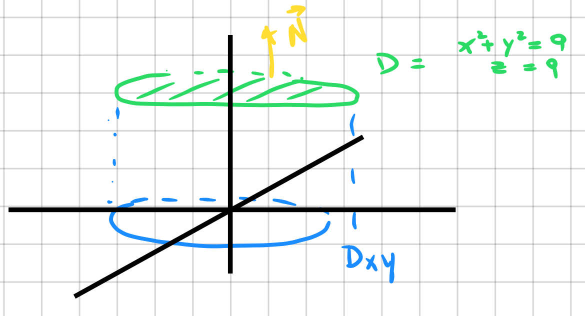
$$\oint_{C^+} \vec{f}\, d\vec{s} = \int_0^{2\pi} f(\vec{g}(t))\cdot \vec{g}'(t)\, dt = \int_0^{2\pi} (3\operatorname{sen}(t)+3\cos(t),\ 6\operatorname{sen}(t),\ 3)(-3\operatorname{sen}(t),\ 3\cos(t),\ 0)\, dt =$$

$$\int_0^{2\pi} -9\operatorname{sen}^2(t) - 9\cos(t)\operatorname{sen}(t) + 18\cos(t)\operatorname{sen}(t)\, dt = -9\left[\int_0^{2\pi} \frac{1}{2} - \cos(2t)\, dt - \frac{1}{2}\int_0^{2\pi} \operatorname{sen}(2t)\cdot dt\right]$$

$$= -9\left(\frac{1}{2}t - \frac{\operatorname{sen}(2t)}{2} + \frac{\cos(2t)}{4}\Bigg|_0^{2\pi}\right) = -9\left[\left(\pi + \frac{1}{4}\right) - \left(\frac{1}{4}\right)\right] = -9\pi$$

ⓒ Calcular circulación con teorema de Stoke $\oint_{\partial D^+} \vec{f}\cdot d\vec{s} = \iint_\Sigma \operatorname{rot}(\vec{f})\cdot \hat{n}\, d\sigma = \iint_{D_{xy}} \operatorname{rot}(\vec{f})(\vec{F})\cdot \frac{\partial F}{\partial x}\times\frac{\partial F}{\partial y}\, dxdy$

- Superficie conveniente $F(x,y) = (x,y,9)$ → plano



$F(x,y) = (x,y,9)\ ,\quad D_{xy} = \{x^2+y^2 \leq 9\}$

- Hallar rotor $\vec{f}$

$$\operatorname{rot}(\vec{f}) = \nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+x & 2y & z \end{vmatrix} = (0,0,1)$$

- Hallar $\frac{\partial F}{\partial x}\times\frac{\partial F}{\partial y}$

$$\frac{\partial F}{\partial x}\times\frac{\partial F}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1)$$

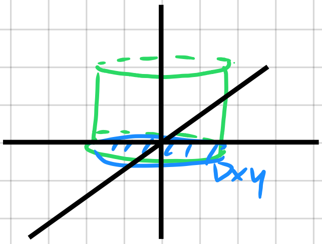
- Hallar $\operatorname{rot}(\vec{f})[\vec{F}(x,y)] = (0,0,1)$

$$\oint_{\partial C^+} \vec{f}\cdot d\vec{s} = \iint_{D_{xy}} (0,0,1)\cdot(0,0,1)\, dxdy = \iint_{D_{xy}} 1\, dxdy = \int_0^3\int_0^{2\pi} r\, d\theta\, dr = \int_0^3 2\pi r\, dr = \pi r^2\Big|_0^3 = 9\pi$$

* Ejercicio 10 $\quad \vec{f}(x,y,z) = (x,\ y+z,\ z)$

Calcular flujo de f por ∂D, $D = \quad x^2+y^2 \le 1, \quad 0 \le z \le 1$

Gráfico D



↳ en vez de calcular por separado, utilizar teorema divergencia

$$\iint_{\partial D^+} \vec{f}\, d\vec{s} = \iiint_D \operatorname{div}(f) \cdot dx\, dy\, dz$$

↳ $\operatorname{div}(f) = \nabla \cdot f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x,\ y+z,\ z) = 1+1+1 = 3$

$$\iint_{\partial D^+} \vec{f}\, d\vec{s} = \iiint_D \operatorname{div}(f) \cdot dx\, dy\, dz = \iint_{D_{xy}} dx\, dy \int_0^1 3\, dz = 3 \iint_{D_{xy}} 1\, dx\, dy$$

$D_{xy} = \{ x^2+y^2 \le 1 \}$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$= 3 \int_0^1 \int_0^{2\pi} r\, d\theta\, dr = 3\int_0^1 2\pi r\, dr = 3\pi r^2 \Big|_0^1 = 3\pi$$